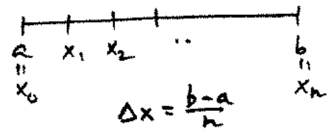
Math 1210-2
Monday Nov. 19

§4.5 Mean value theorem for integrals.
Symmetry time savers

<u>Definition</u>: If f is integrable on the interval $[a,b]$ then the <u>average value of f on $[a,b]$</u> is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

<u>Reason for def</u>: Partition $[a,b]$ into n equal length subintervals



The average of $f(x_1), f(x_2), \ldots f(x_n)$ is

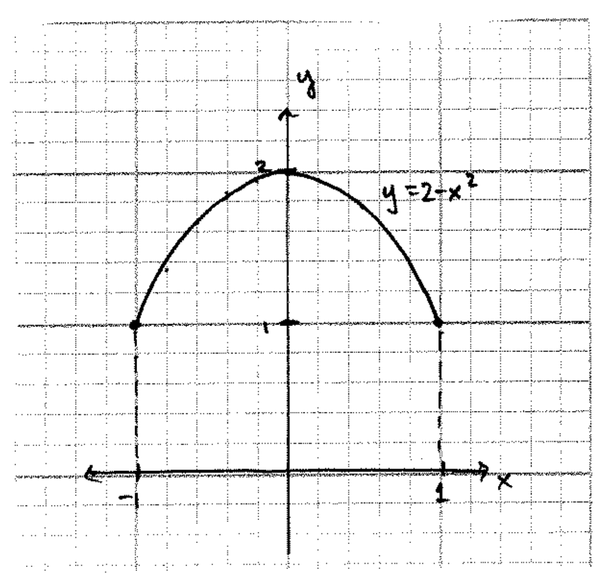
$\frac{1}{n}\sum_{i=1}^{n} f(x_i)$. The "average value of f" should be the limit of this average, as $n \to \infty$.

But,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) = \frac{1}{b-a}\underbrace{\sum_{i=1}^{n} f(x_i)\Delta x_i}_{\text{as } n\to\infty \text{ this converges to } \int_a^b f(x)dx}$$

so the limit of these actual averages is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

<u>Exercise 1</u>: Find the average value of $f(x) = 2 - x^2$ on the interval $[-1,1]$.
Interpret this value as the height of a rectangle with base $-1 \le x \le 1$ on the x-axis, where the rectangle has the same area as is under the graph.



<u>Exercise 2</u> Let $v(t)$ be the velocity of a particle moving along a number line with position $s(t)$, i.e. $s'(t) = v(t)$. What is

(a) The average velocity between $t=a$ and $t=b$?
(as we used to define it as a prelude to taking a limit to get instantaneous velocity)

(b) The average value of the velocity for $a \le t \le b$, as defined on page 1?
(Are we lucky, or what?!)

<u>Mean Value Theorem for Integrals</u>: If $f(x)$ is continuous on $[a,b]$, then there is at least one c, $a<c<b$, at which $f(c)$ equals the average value of f, i.e.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

proof: This is old MVT in disguise: $\frac{1}{b-a}\int_a^b f(x)\,dx \overset{\text{FTC}}{=} \frac{1}{b-a}\left(F(b)-F(a)\right) \qquad (F'=f)$

$= F'(c) \qquad$ old MVT, some $a<c<b$.

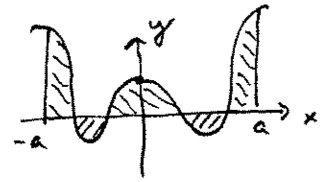
$= f(c)$

<u>Exercise 3</u>: Verify MVT for Integrals, for Exercise 1 example.

Symmetry shortcuts

(1) If $f(x)$ is **even**, $(f(-x)=f(x))$

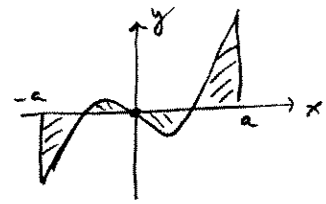
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



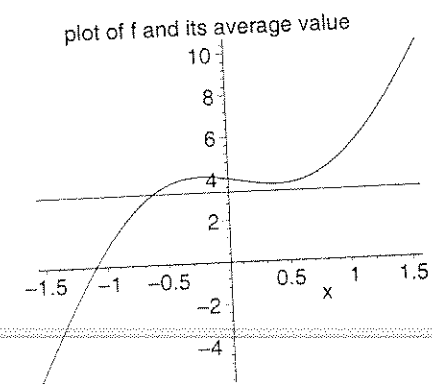
check by substitution:

(2) If $f(x)$ is **odd**, $(f(-x)=-f(x))$

$$\int_{-a}^{a} f(x)\,dx = 0$$



check:



Exercise 4 : Use symmetry to compute

a) $$\int_{-\pi/2}^{\pi/2} x^3 + x^2 + x^2\sin x - x\cos(2x) + 4\cos x \; dx$$

b) The average value of $f(x)$, the integrand in (a), for $-\pi/2 \le x \le \pi/2$.

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you **don't** want to use:

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2\cos(x) + 2x\sin(x) + C$$

$$\int x\cos(2x)\,dx = \frac{1}{4}\cos(2x) + \frac{1}{2}x\sin(2x) + C$$

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3rd Midterm score distribution
n=99