Warmup:

What's an antiderivative?

What's an indefinite integral?

What's a definite integral?

On Friday we proved the 2nd Fund Thm. Calc.

\[ f \text{ continuous on } [a,b] \]
\[ F \text{ on } (a,b) \]
\[ F' = f \text{ on } (a,b) \]

Then

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i \]

actually equals

\[ F(b) - F(a). \]

Exercise 1: Interpret FTC2 in case \( f = v(t) = \text{velocity at time } t \)
\( F = s(t) = \text{position at time } t \)

(Because in this case FTC2 seems quite natural!)
This idea always works, to recover the net change in a quantity, if you know how it's been changing:

Exercise 2: Water is leaking out of a tank at a rate of

\[ V'(t) = 11 - 1.1t \text{ gal/hour, where } V(t) = \text{water volume}. \]

How much water leaks out between \( t = 1 \) and \( t = 3 \) hours?

Properties of the Definite Integral

Page 3 Friday.

Also, inequality property:

(4) If \( f(x) \leq g(x) \) on \([a,b]\), then

\[ \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \]

Because, for a given partition \( P \) and choice of sample points,

\[ \sum_{i=1}^n f(x_i^*) \Delta x_i \leq \sum_{i=1}^n g(x_i^*) \Delta x \]

Now, take limit as \( ||P|| \to 0 \). \( \blacksquare \)
So, FTC2 is great, but what is you can’t find an antiderivative?

*\( \int \frac{1}{t} \, dt = ? \)

*\( \int \frac{t^3 \cos(t^2+1)}{t^2+1} \, dt = ? \)

This is where FTC1 enters the picture - it shows that every continuous function on \([a, b]\) **does** have an antiderivative.

**Theorem (First Fundamental Theorem of Calculus).**

Let \( f(x) \) be continuous on \([a, b]\).

Define the **accumulation** (or area?) function

\[
A(x) = \int_a^x f(t) \, dt
\]

Then \( A(x) \) is an antiderivative of \( f(x) \),

\[
A'(x) = f(x)
\]

**Exercise 3** Verify FTC1 in this case, where you can compute \( A(x) \) explicitly:

3a) \( A(x) = \int_1^x t^2 \, dt \)

3b) \( A(x) = \int_a^x f(t) \, dt \), if you already know an antiderivative \( F(x) \).

**Remark:** \( A(x) = \int_1^x \frac{1}{t} \, dt \) is a very special accumulation function (and is our missing antiderivative of \( \frac{1}{x} \)).

In fact, this \( A(x) = \ln(x) \) is the natural logarithm function!

See §6.1!
proof of FTC I:

\[ A(x) = \int_a^x f(t) \, dt \]

\[ A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \]

\[(*) = \lim_{h \to 0} \frac{1}{h} \left\{ \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt \right\} \]

We will consider

\[ \lim_{h \to 0^+} \]

i.e., \( h > 0 \). The other case is analogous.

For \( h > 0 \),

\[ \int_a^{x+h} f(t) \, dt = \int_a^x f(t) \, dt + \int_x^{x+h} f(t) \, dt \]

So

\[(*) = \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t) \, dt \]

\[ m = \frac{1}{h} m h = \int_x^{x+h} m(t) \, dt \leq \int_x^{x+h} f(t) \, dt \leq \int_x^{x+h} M \, dt = \int_x^{x+h} \frac{1}{h} M h = M \]

if \( m = \) minimum value of \( f(t) \),

\[ x \leq t \leq x+h \]

\[ m \leq \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq M \]

Now "squeeze" theorem!, since \( f \) is continuous at \( x \), both \( m \) and \( M \) approach \( f(x) \) as \( h \to 0^+ \), i.e.

\[ f(x) = \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq f(x) \]