Math 1210-2
Monday Nov. 12

- Carnival Wednesday
- Exam Friday – covers thru §4.

Warmup:

What's an antiderivative?

What's an indefinite integral?

What's a definite integral?

On Friday we proved the 2nd Fdl Thm. Calc:

f continuous on $[a,b]$
F " " "
$F' = f$ on (a,b)

Then
$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i$$
actually equals
$$F(b) - F(a).$$

Exercise 1: Interpret FTC2 in case $f = v(t)$ = velocity at time t
$F = s(t)$ = position at time t

(Because in this case FTC2 seems quite natural!)

This idea always works, to recover the net change in a quantity, if you know how it's been changing:

Exercise 2: Water is leaking out of a tank at a rate of

$$V'(t) = 11 - 1.1t \quad \text{gal/hour}, \quad \text{where } V(t) = \text{water volume}.$$

How much water leaks out between $t=1$ and $t=3$ hours?

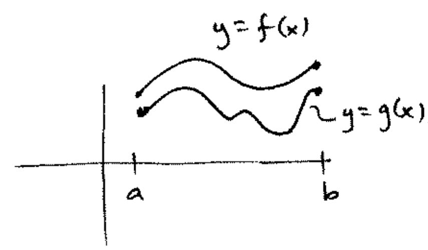
Properties of the Definite Integral

Page 3 Friday.

Also, inequality property:

(4) If $f(x) \le g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



Because, for a given partition P and choice of sample points,

$$\sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i \le \sum_{i=1}^{n} g(\bar{x}_i)\,\Delta x$$

Now, take limit as $\|P\| \to 0$. ■

So, FTC2 is great, but what is you can't find an antiderivative?

e.g.

$$\int \frac{1}{t}\, dt = ?$$

$$\int \frac{t^3 \cos(\sqrt{t+1})}{t^2+1}\, dt = ?$$

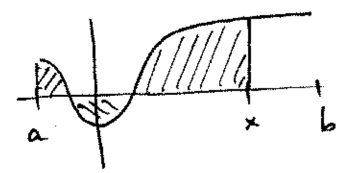
This is where FTC1 enters the picture - it shows that every continuous function on $[a,b]$ does have an antiderivative.

<u>Theorem</u> (First Fundamental Theorem of Calculus).

Let $f(x)$ be continuous on $[a,b]$.

Define the <u>accumulation</u> (or area?) <u>function</u>

$$A(x) = \int_a^x f(t)\,dt$$



Then $A(x)$ is <u>an</u> antiderivative of $f(x)$,

$$A'(x) = f(x)$$

<u>Exercise 3</u> Verify FTC1 in this case, where you can compute $A(x)$ explicitly:

3a) $A(x) = \int_1^x t^2\,dt$

3b) $A(x) = \int_a^x f(t)\,dt$, if you already know an antiderivative $F(x)$.

<u>Remark</u>: $A(x) = \int_1^x \frac{1}{t}\,dt$ is a very special accumulation function (and is our missing antiderivative of $\frac{1}{x}$).

In fact, this $A(x) = \ln(x)$ is the natural logarithm function! See §6.1!

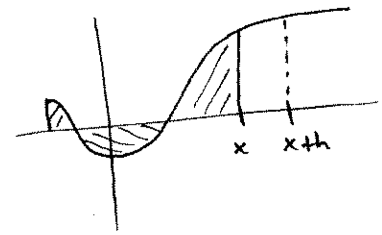
proof of FTC I :

$$A(x) = \int_a^x f(t)\,dt$$

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

$$(*) \quad = \lim_{h\to 0} \frac{1}{h}\left\{ \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right\}$$

We will consider
$\lim_{h\to 0^+}$
i.e., $h>0$. The other case is analogous.



For $h>0$, $\int_a^{x+h} f(t)\,dt = \int_a^x \text{“}-\text{”} + \int_x^{x+h} \text{“}-\text{”}$

So

$$(*) = \lim_{h\to 0^+} \frac{1}{h} \int_x^{x+h} f(t)\,dt .$$

$$m = \frac{1}{h} mh = \frac{1}{h}\int_x^{x+h} m\,dt \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le \frac{1}{h}\int_x^{x+h} M\,dt = \frac{1}{h} Mh = M$$

if m = minimum value of $f(t)$, $x \le t \le x+h$

if M = maximum value of $f(t)$, $x \le t \le x+h$

$$m \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M$$

Now "squeeze" theorem!; since f is continuous at x, both m and M approach $f(x)$ as $h\to 0^+$, i.e.

$$f(x) \le \lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x)$$