Name Solutions

Student I.D. ______________________

Math 1210-2
Final Exam
December 11, 2007

Please show all work for full credit. This exam is closed book, closed note, closed calculator, except for the single customized 4 by 6 inch index card you have been allowed to bring. (Two Varberg formula pages are at the end of the exam.) There are 150 points possible, as indicated below and in the exam. You have two hours to complete the exam, so apportion your time accordingly. Good Luck!!

	Score	POSSIBLE
1		20
2		15
3		15
4		15
5		20
6		15
7		15
8		25
9		10
TOTAL		150

1) Compute the following limits.

1a) $\lim\limits_{x\to 1} \dfrac{x^2+4x-5}{x^2-x} = \lim\limits_{x\to 1} \dfrac{\cancel{(x-1)}(x+5)}{x\cancel{(x-1)}} = \dfrac{6}{1} = \boxed{6}$ (5 points)

$\frac{0}{0}$

1b) $\lim\limits_{x\to 0^-} \dfrac{2x}{|x|} = \lim\limits_{x\to 0^-} \dfrac{2x}{-x} = \lim\limits_{x\to 0^-} -2 = \boxed{-2}$ (5 points)

0^-

1c) $\lim\limits_{t\to\infty} \dfrac{3t^2-5t+6}{t+7t^2} = \lim\limits_{t\to\infty} \dfrac{\cancel{t^2}(3-5/t+6/t^2)}{\cancel{t^2}(1/t+7)} = \boxed{\dfrac{3}{7}}$ (5 points)

1d) $\lim\limits_{h\to 0} \dfrac{\sin(\overset{3(x+h)}{3x+3h})-\sin(3x)}{h}$

(Hint: If you recognize this as the derivative of a certain function you can deduce the answer without computing any limits.)

(5 points)

$= D_x \sin 3x = \boxed{(\cos 3x)3}$

long way uses addition angle formulas

$= \lim\limits_{h\to 0} \dfrac{\sin 3x \cos 3h + \cos 3x \sin 3h - \sin 3x}{h}$

$= \lim\limits_{h\to 0} \sin 3x \left(\dfrac{\cos 3h - 1}{h}\right) + \cos 3x \dfrac{\sin 3h}{h}$

$= \sin 3x \lim\limits_{h\to 0} 3\left(\underset{\downarrow\ 0}{\dfrac{\cos 3h-1}{3h}}\right) + \cos 3x \lim\limits_{h\to 0} 3\underset{\downarrow\ 1}{\dfrac{\sin 3h}{3h}} = \boxed{3\cos 3x}$

2) Compute the following derivatives:

2a)

$$-8x^{-3}$$

$$D_x\left(4x^3 - \frac{8}{x^3} + 16\right)$$

(5 points)

$$= 4\cdot 3x^2 - 8(-3x^{-4}) + 0$$

$$\left(= 12x^2 + 24x^{-4} \right)$$

2b)

$$D_t\left[\frac{\cos(3t^2)}{\sqrt{4t+3}}\right]$$

(5 points)

$$= \frac{(-\sin 3t^2)(6t)\sqrt{4t+3} - \cos(3t^2)\,\frac{1}{2}(4t+3)^{-1/2}\,4}{4t+3}$$

$$\left(= \frac{-6t\sin 3t^2}{\sqrt{4t+3}} - \frac{2\cos 3t^2}{(4t+3)^{3/2}} \right)$$

2c) Suppose f(1)=3, f′(1)=-5, f(2)=-2, f′(2)=3, g(1)=2, g′(1)=-4. Compute the derivative of the function

$$8\,f(x) - [g(x)]^3 + f(g(x))$$

at x=1.

(5 points)

$$D_x(\ ) = 8f'(x) - 3g(x)^2 g'(x) + f'(g(x))\,g'(x)$$

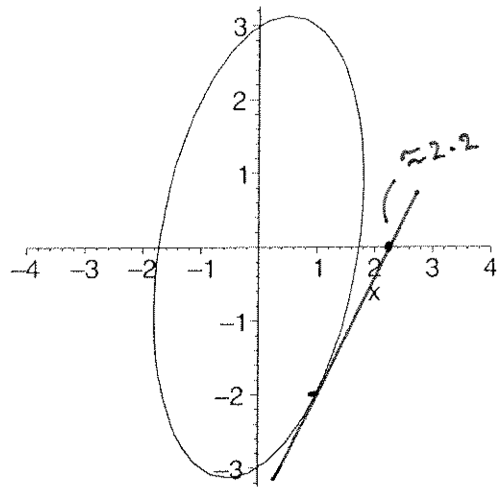
$$@\ x=1, \quad = 8(-5) - 3(4)(-4) + \underbrace{f'(g(1))}_{3}(-4)$$

$$= -40 + 48 - 12$$

$$\boxed{= -4}$$

3a) Use the plot below to carefully sketch the line tangent to the graph of $3x^2 - xy + y^2 = 9$, at the point $(x, y) = (1, -2)$. Suggestion: pull off the Varberg formula sheets and fold one of them for a ruler.

(3 points)



3b) Use the point (1,-2) and the x-intercept of your tangent line sketch above, to estimate this line's slope.

(4 points)

$m \approx \frac{0 - -2}{2.2 - 1} = \frac{2}{1.2} = \frac{1}{.6} = \frac{10}{6} = \frac{5}{3} \approx 1.67$

3c) Use implicit differentiation to find the exact slope of the tangent line to the curve above, at the point (1,-2), and then write an exact equation for the tangent line you sketched in part (3a).

(5 points)

$3x^2 - xy + y^2 = 9$

D_x: $6x - y - xy' + 2yy' = 0$

$y'(2y - x) = y - 6x$

$y' = \frac{y - 6x}{2y - x}$; @ $(1,-2)$ $y' = \frac{-2 - 6}{-4 - 1} = \frac{8}{5} = \boxed{1.6}$

tang line

$(y+2) = \frac{8}{5}(x-1)$

3d) What is the (exact) x-intercept of the tangent line you found above?

(3 points)

if $y = 0$, then

$2 = \frac{8}{5}(x-1)$

$2\left(\frac{5}{8}\right) = x - 1$

$x = 1 + \frac{5}{4} = \boxed{2.25}$

4) Compute the following integrals:

4a) $\int_{-2}^{2} 5x^3 + 2x^2\,dx \quad = 2\int_0^2 2x^2\,dx = 2\cdot\frac{2}{3}x^3\Big]_0^2 = \boxed{\frac{32}{3}}$

odd even

symmetric interval

(5 points)

4b) $\int_1^8 \frac{10}{\sqrt{3x+1}}\,dx$

$u = 3x+1$

$du = 3\,dx$

$= \frac{10}{3}\int_1^{10} \frac{1}{\sqrt{3x+1}}\,3\,dx$

(5 points)

$x=1$
$u=4$

$x=8$
$u=25$

$= \frac{10}{3}\int_4^{25} u^{-1/2}\,du = \frac{10}{3}\,2u^{1/2}\Big]_4^{25}$

$= \frac{20}{3}(5-2) = \boxed{20}$

4c) $\int_0^{\frac{\pi}{4}} \frac{\sin(2t)}{(3+\cos(2t))^2}\,dt$

(5 points)

$u = 3 + \cos 2t$

$du = -2\sin 2t\,dt$

$= -\frac{1}{2}\int_0^{\pi/4} \frac{-2\sin 2t\,dt}{(3+\cos 2t)^2} = -\frac{1}{2}\int_4^3 \underbrace{\frac{du}{u^2}}_{u^{-2}du} = -\frac{1}{2}\left(\frac{u^{-1}}{-1}\right)\Big]_4^3 = \frac{1}{2}\frac{1}{u}\Big]_4^3$

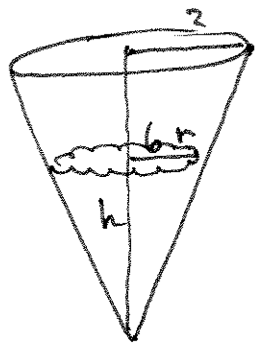
$t=0 \quad t=\pi/4$
$u=4 \quad u=3$

$= \frac{1}{2}\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{\frac{1}{12}}$

$\boxed{= \frac{1}{24}}$

5a) A mountain cabin has a drinking-water cistern, shaped like an upside-down cone. The depth is 6 feet, and the circular top has a radius of 2 feet. The cistern is filled with water from a spring. After being totally emptied, it is being refilled. When the depth of the water is 3 feet it is increasing at a rate of 3 inches per minute. How fast is water flowing into the cistern, at that instant, assuming no water is flowing out at the same time?

(15 points)

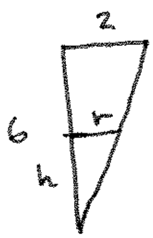


When $h=3$, $h'(t) = 3''/\text{min} = \frac{1}{4}$ ft/min

Find $V'(t)$, (V = volume) at that instant.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{h} = \frac{2}{6} = \frac{1}{3} \;;\; r = \frac{1}{3}h$$



so $V = \frac{1}{27}\pi h^3$

$$V'(t) = \frac{1}{27}\pi\, 3h^2 h' = \frac{\pi}{9} h^2 h'$$

@ $h=3, h'=.25$, $V'(t) = \frac{\pi}{\cancel{9}}(\cancel{9})(.25) = .25\pi$ ft³/min

5b) Assuming the inflow rate remains constant and that no one is using water from the cistern, how much later with the cistern be completely refilled?

(5 points)

still need to fill a volume of

$$\frac{1}{3}\pi(2^2)6 - \frac{1}{3}\pi(1^2)(3) \qquad (\text{when } h=3,\ r=\tfrac{1}{3}h=1.$$

$$= \frac{1}{3}\pi(24-3)$$

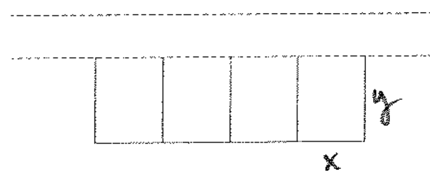
$$= \frac{21}{3}\pi$$

$$= 7\pi \text{ ft}^3.$$

(rate)(time) = amount, so $T = \frac{7\pi}{.25\pi} = \frac{7}{.25}\frac{(4)}{(4)} = \boxed{28 \text{ minutes}}$

$.25\pi \quad T \quad 7\pi$

6) Farmer Sally wishes to test four new strains of table corn, as well as advertise her farming abilities. She will fence off four adjacent and congruent rectangular plots, one for each variety of corn, using the road as the "northern" (fenceless) border for each plot, as indicated in the diagram below:



Farmer Sally requires that each of the four plots have area 2000 square feet. What dimensions for the individual plots will satisfy that requirement while at the same time using the least total length of fencing?

6a) Find the answer to the question above using Calculus.

(10 points)

$xy = 2000$

minimize $P = 4x + 5y$

$y = \frac{2000}{x}$

$P(x) = 4x + 5\left(\frac{2000}{x}\right)$

$= 4x + \frac{10^4}{x}$

$P'(x) = 4 + 10^4(-x^{-2}) = 0$

$4 = \frac{10^4}{x^2}$

$x^2 = \frac{10^4}{4} = 2500;$ $\boxed{x = 50 \quad (x > 0) \quad y = 2000/x = 40}$

6b) Justify why your answer must be correct, using concepts related ~~to local and global~~ extrema which we've discussed in this class.

(5 points)

(1) the domain for x is $x > 0$

$P'(x) = 4 - 10^4 x^{-2}$

$P''(x) = 2 \cdot 10^4 x^{-3} > 0$ for $x > 0$

so function is concave up on $0 < x < \infty$ and its stationary point yields a global minimum value for perimeter.

or (2) $P'(x) = 4 - \frac{10^4}{x^2}$

has sign chart

$- - - 0 + + +$

dec 50 inc.

so $P(50)$ is global min

7a) Sketch the region between the graphs of $y = x^2 - 4$ and $y = x + 2$. Make sure to label the curves and the points where they intersect.

(5 points)

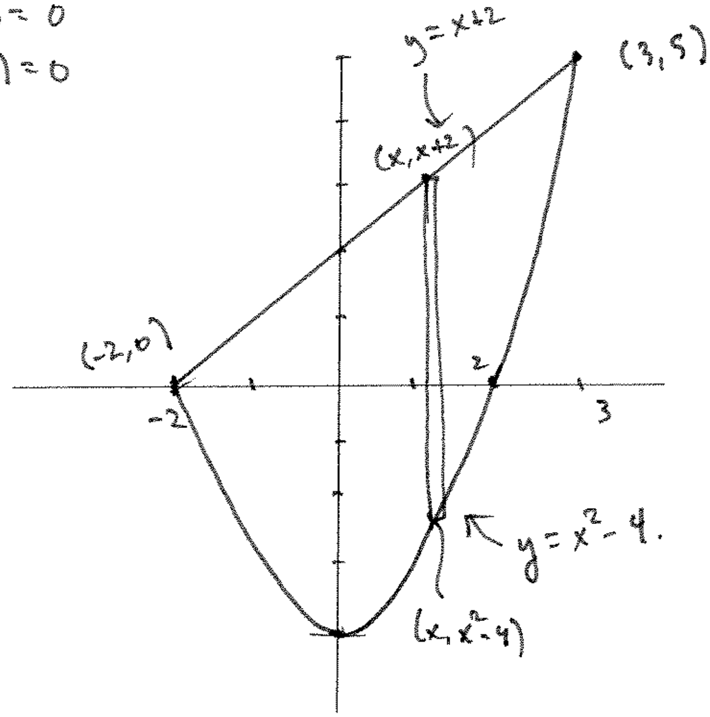
cross at $x^2 - 4 = x + 2$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = -2, 3$

$(-2,0), (3,5)$



7b) Find the area of the region described in ~~(6a)~~ 7a.

(10 points)

$$\Delta A \simeq [(x+2) - (x^2-4)]\,\Delta x$$

$$A = \int_{-2}^{3} x+2-x^2+4\,dx = \int_{-2}^{3} -x^2+x+6\,dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 6x\Big]_{-2}^{3}$$

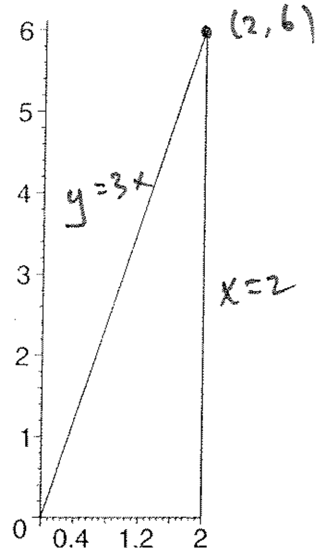
$$= -9 + \frac{9}{2} + 18 - \Big[-\frac{8}{3} + \overbrace{2-12}^{-10}\Big]$$

$$= 9 + \frac{9}{2} - \frac{8}{3} + 10$$

$$= 9 + 4\tfrac{1}{2} - 2\tfrac{2}{3} + 10$$

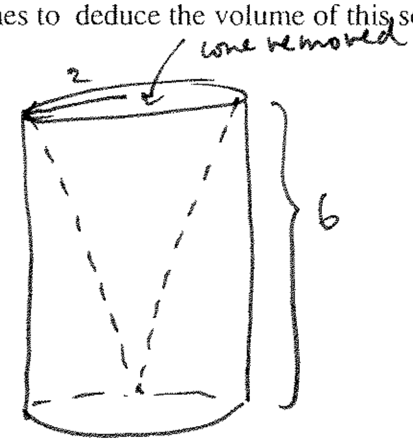
$$= 21 + \underbrace{\tfrac{1}{2} - \tfrac{2}{3}}_{-1/6} = \boxed{20\tfrac{5}{6}} \quad \left(= \frac{125}{6}\right).$$

8) Consider the triangle in the first quadrant with edges along the x-axis, the line $y = 3x$ and the vertical line $x = 2$. Because you will be doing several computations related to this triangular region, here is a picture of it just to make sure there is no confusion:



8a) If this triangular region is rotated about the y-axis a solid is created which can be thought of as a vertical cylinder, from which an upside-down cone has been removed. Use volume formulas for cylinders and cones to deduce the volume of this solid of revolution.

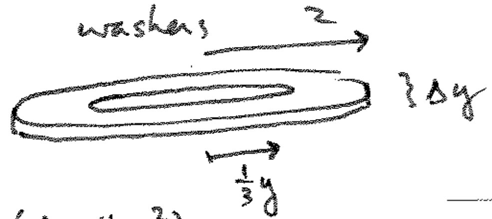
(5 points)

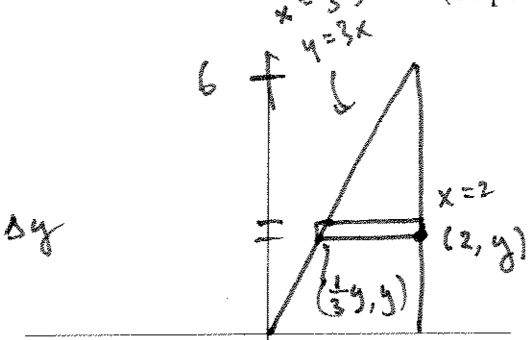


$$V = \text{vol(cylinder)} - \text{vol(cone)}$$
$$= \pi r^2 h - \frac{1}{3}\pi r^2 h$$
$$= \frac{2}{3}\pi r^2 h$$
$$= \frac{2}{3}\pi (4)(6)$$
$$\boxed{= 16\pi}$$

8b) Now use the method of planar slabs to recompute the volume of the same solid of revolution.

(10 points)





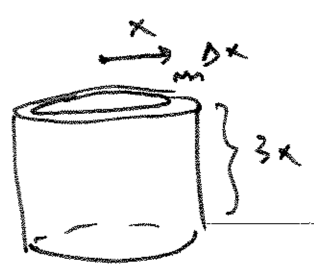
$$\Delta V \approx \pi \left(4 - \left(\tfrac{1}{3}y\right)^2\right) \Delta y$$

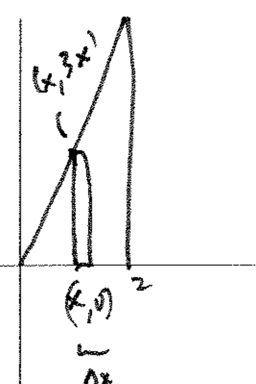
$$V = \int_0^6 \pi \left(4 - \frac{y^2}{9}\right) dy = \pi \left(4y - \frac{y^3}{27}\right]_0^6$$

$$= \pi\left(24 - \frac{\cancel{3^3}\, 2^3}{\cancel{27}}\right) = \pi(24 - 8) = \boxed{16\pi}$$

8c) Finally, recompute the volume of revolution, this time with cylindrical shells. If you do all of your work correctly, your answers to 8 a,b,c will all agree.

(10 points)





$$\Delta V \approx 2\pi x (3x) \Delta x$$

$$V = \int_0^2 2\pi (3x^2)\, dx = 2\pi x^3 \Big]_0^2$$

$$\boxed{= 16\pi}$$

9) Find the centroid for the triangle having vertices (0,0), (2,0), and (2,6), and assuming the density (mass per unit area) is a constant $\delta = 1$. This is the same triangle you studied in problem 8. The most straightforward way to do this problem is to compute the appropriate ratios of moments to mass, although you could alternately use Pappus' Theorem.

(10 points)

Pappus: $A(2\pi\bar{x})$ = vol of revolution abt x-axis

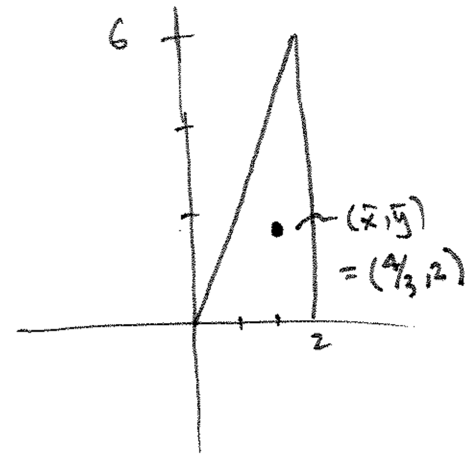
$= 16\pi$

A = area of $\Delta = \frac{1}{2}(2)(6) = 6$

So

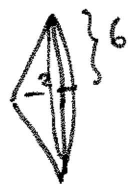
$12\pi\bar{x} = 16\pi$

$\boxed{\bar{x} = 4/3}$



$A(2\pi\bar{y})$ = vol of rev. abt y-axis

= cone

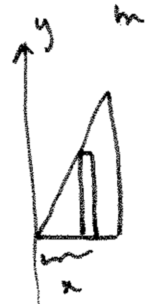


$12\pi\bar{y} = \frac{1}{3}\pi(36)\cdot 2 = 24\pi$

$\boxed{\bar{y} = 2}$

or, $\bar{x} = \dfrac{M_y}{m} = \dfrac{\int_0^2 3x^2\,dx}{6} = \dfrac{x^3\Big]_0^2}{6} = \dfrac{8}{6} = \boxed{4/3}$

m = area since $\delta = 1$



$\Delta M_y = (3x\,\Delta x)x$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{\frac{1}{2}\int_0^2 f^2(x) - g^2(x)\,dx}{6}$

$= \dfrac{\frac{1}{2}\int_0^2 9x^2\,dx}{6}$

$= \dfrac{\frac{1}{2}\,3x^3\Big]_0^2}{6}$

$= \dfrac{12}{6} = \boxed{2}$