Math 1210-2
Wednesday Dec. 5.

§5.6 Moments & center of mass
(last day of new math - Friday is review.)
Saturday problem session 1:30-3:00
Final exam (inclusive) on <u>Tuesday Dec 11, 8-10 a.m</u>, here (JWB 335)

The balance point for masses distributed along a line is the $\bar{x}$ which yields a zero "moment":

$$\sum_{i=1}^{n} (x_i - \bar{x}) m_i = 0$$

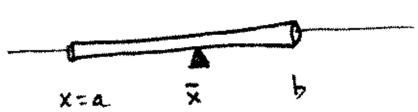
$$\sum x_i m_i - \bar{x} \sum m_i = 0$$

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} := \frac{M}{m}}$$

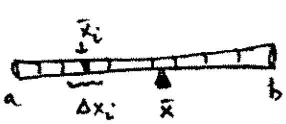
← Moment wrt $x=0$.
← total mass

can be motivated as pivot point at which it takes no net work to rotate the configuration, e.g. working against gravity

Now consider a rod with varying density (mass/length) $\delta(x)$, $a \le x \le b$.



partition, approximate, limit:



$$\bar{x} \approx \frac{\sum_{i=1}^{n} \bar{x}_i \big(\overbrace{\delta(\bar{x}_i)\Delta x_i}^{\Delta m_i} \big)}{\sum \underbrace{\delta(\bar{x}_i)\Delta x_i}_{\Delta m_i}}$$

limit!

$$\boxed{\bar{x} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx} := \frac{M}{m}}$$

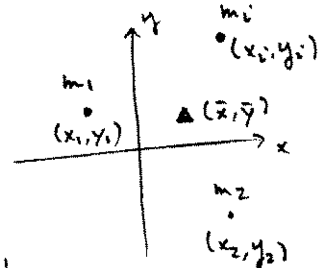
moment ↓ (M)
↑ total mass (m)

<u>Exercise 1</u> a wire of varying diameter has density

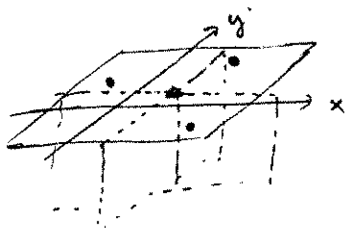
$$\delta(x) = .04\,x^3 \quad gm/cm$$
$$0 \le x \le 10 \text{ cm}.$$

(a) Find its total mass
(b) Find its center of mass

point masses in the x-y plane :



<u>perspective</u>



If the points balance on a vertical plane along $x = \bar{x}$, and on a vertical plane along $y = \bar{y}$, then they will balance at the point $(\bar{x}, \bar{y})$

signed distance to x-axis

M_x = moment with respect to x-axis $= \sum_{i=1}^{n} m_i y_i$

M_y = " " " y-axis $= \sum_{i=1}^{n} m_i x_i$ signed dist to y-axis

$m = \sum_{i=1}^{n} m_i$ total mass

$\bar{x} = \dfrac{M_y}{m} = \dfrac{\sum m_i x_i}{\sum m_i}$ (looks like old formula, this way)

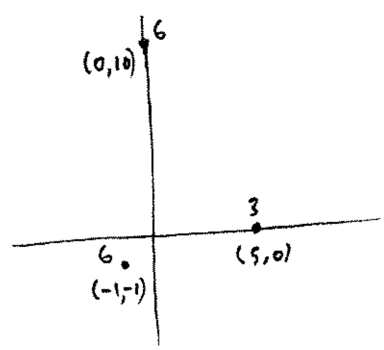
$\bar{y} = \dfrac{M_x}{m} = \dfrac{\sum m_i y_i}{\sum m_i}$

<u>Exercise 2</u> $m_1 = 3$ at $(5,0)$

$m_2 = 1$ at $(0,10)$

$m_3 = 6$ at $(-1,-1)$

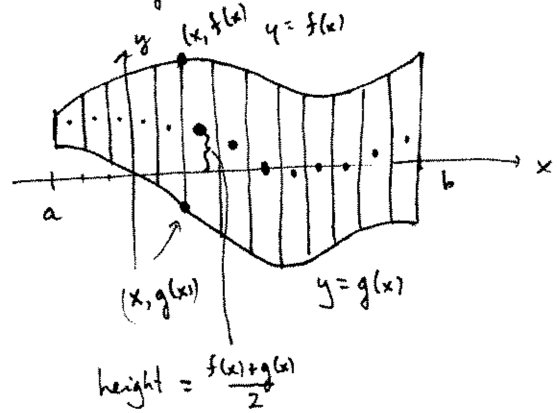
Find $(\bar{x}, \bar{y})$



In multivariable Calculus (Math 2210) you learn how to find moments and centers of mass for 2-d & 3-d objects with varying mass-density functions.

Using single variable calc we can handle lamina, (thin planar sheets), with constant density δ mass/area.

Consider a region:



this one's tricky!

ht of ctr of mass of vertical strip

$$\Delta m \approx \delta\,(f(x)-g(x))\,\Delta x \qquad \Delta M_y \approx x(\Delta m) \approx x\,\delta\,(f(x)-g(x))\,\Delta x$$

$$\Delta M_x \approx \frac{(f(x)+g(x))}{2}\,\Delta m = \frac{f(x)+g(x)}{2}\,\delta\,(f(x)-g(x))\Delta x = \frac{\delta}{2}(f^2(x)-g^2(x))\Delta x$$

$$m = \int_a^b \delta\,(f(x)-g(x))\,dx, \qquad M_y = \int_a^b \delta\, x\,(f(x)-g(x))\,dx,$$

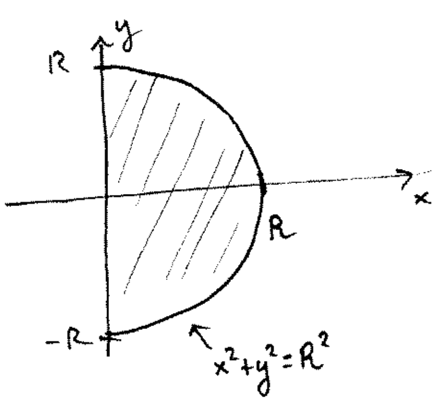
$$M_x = \int_a^b \frac{\delta}{2}\,(f^2(x)-g^2(x))\,dx$$

for $\bar{x}$ & $\bar{y}$ the δ's cancel out:

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x\,(f(x)-g(x))\,dx}{\int_a^b f(x)-g(x)\,dx} \leftarrow \text{area! } A \qquad \bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}(f^2(x)-g^2(x))\,dx}{A}$$

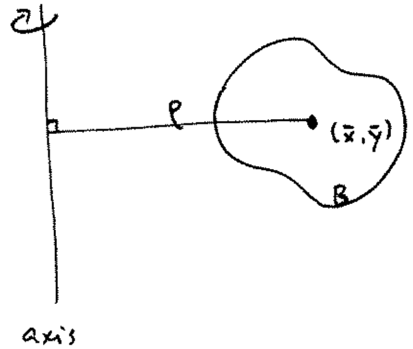
Exercise 3 : Find the center of mass of a half-disk lamina:

$(\bar{x}, \bar{y})$ is called the centroid



Cool fact:

<u>Pappus' Theorem</u> (circa A.D. 300 - those amazing Greeks!)



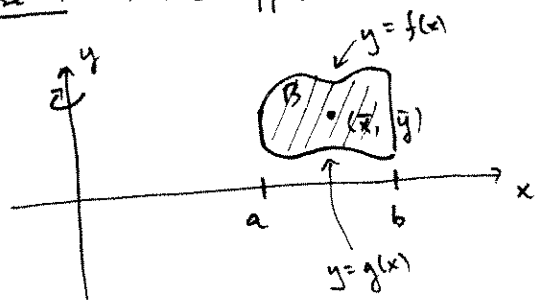
Rotate region B about an axis on one side of it.

Let ρ be the distance from axis to centroid
Let B have area A.
Then the volume of revolution is

$$V = (2\pi\rho) A$$

(like the volume of a <u>straight</u> cylinder of length $2\pi\rho$ & cross-section area A!)

<u>Exercise 4</u> Prove Pappus' Thm in case:



Use cylindrical shells for the volume.

<u>Exercise 5</u>: What is the volume of this doughnut?:
(rotate circle abt y-axis)