We need to finish discussing curve lengths and areas of surfaces, from Friday.

Then, §5.5 Work:

In physics, the work done to move an object a distance $d$ against a force $F$ is

$$W = F \cdot d$$

Example: to lift a mass $m$ a height $h$ takes $W = (mg)h$ work.

Units: in metric system

- $F$ units: $1 \text{ kg m/sec}^2 = 1 \text{ Newton}$
- $W$ units: $1 \text{ Newton-meter} = 1 \text{ joule}$

In English system

- $F$ units: pounds ($1 \text{ pound} = 1 \text{ slug ft/sec}^2$)
- $W$ units: $1 \text{ foot-pound}$

In a closed physical system, work converts kinetic energy to potential energy, $KE$ and $PE$, and the total energy $= KE + PE$ is constant.

Example: For a thrown object (neglecting friction),

$$KE + PE = \frac{1}{2}mv^2 + mgh \quad \text{is constant}$$

Check: take $\frac{d}{dt}$ (total energy)

$$= mvv'(t) + mgv$$

$$= v(mv' + g)$$

$$= 0 \quad \text{iff} \quad h''(t) = -g, \quad \text{Newton's Law.}$$

Forces can change depending on location.

E.g., Earth's gravitational attraction on an object is really

$$|F| = \frac{GMm}{r^2}$$

$G$ = universal constant
$M$ = mass of earth
$m$ = object mass
$r$ = distance to center of Earth

So in fact

$$\frac{GM}{R^2} = g = 9.8 \text{ m/sec}^2$$

For $R$ = radius of Earth

So,

$$|F| = mg\left(\frac{R^2}{r^2}\right)$$
Work computation for space-varying force:

\[ F = F(x) \]

\[ a \rightarrow \hat{x}_i \rightarrow b \]

Work done to move object from \( a \) to \( b \)

\[ W \approx \sum_{i=1}^{n} F(\hat{x}_i) \Delta x_i \]

So

\[ W = \int_{a}^{b} F(x) \, dx \]

**Exercise 1:** How much work must be done on an object to move it from the surface of the Earth to \( \infty \), neglecting friction?

\[ W = \int_{R}^{\infty} mg \left( \frac{R^2}{r^2} \right) \, dr = \]

**Exercise 2:** What should the initial speed of an object be to barely escape the Earth?

Use \( KE + PE = \text{const} \) to recompute escape velocity \( v_e = \sqrt{2gR} \approx 6.98 \text{ m/s} \).

(want \( v \to 0 \) as \( r \to \infty \), so at \( \infty \) \( KE = 0 \)).

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**SPRINGS**

\[ \text{spring at rest} \]

\[ \text{stretched spring} \quad \text{(compressed if } x < 0) \]

Hooke's Law

\[ F(x) = kx \quad k: \text{spring constant} \]

is really just the tangent line approximation ("linear approx") to a differentiable force function which satisfies \( F(0) = 0 \).
Exercise 3. If natural length of a spring is 0.2 m
and 12 N force is required to stretch it an additional 0.04 m

1) Find the spring constant k
2) How much work is done in stretching the spring from its
natural length to 0.3 m?

Spring dynamics [and springs are everywhere in Science]
set total energy = 0 for mass-spring at rest.

\[ E = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 \]

\[ T = \frac{1}{2} kx^2 \]

\[ W = \int_0^x kx \, ds = \frac{1}{2} kx^2 \]

\[ x'(t) = v \]

\[ \frac{dx}{dt} = \frac{d}{dt} \left( x(t) \right) = x'(t) \left[ kx + mx''(t) \right] \]

\[ \frac{dE}{dt} = 0 = kx'x'(t) + mv'(t)x''(t) \]

\[ \Rightarrow \frac{d}{dt} \left( x'(t) \right) = \frac{d}{dt} \left( x'(t) \right) \left[ kx + mx''(t) \right] \]

\[ \text{deduce} \quad x''(t) = -\frac{k}{m} x(t) \]

[could get this directly from Newton too]

Solution to
\[ x''(t) = -\omega_0^2 x(t) \]

is
\[ x(t) = C \cos(\omega_0 t - \phi) \]

\[ = A \cos \omega_0 t + B \sin \omega_0 t \]

and this is the primary reason
for the importance of trigonometry
is science
Exercise 4
Consider a conical water tank, height = 10 feet, \( r = 4 \) feet full of water.

How much work must be done to pump all of the water in the tank to a height 5 feet above the top of the tank? Water weighs 62.4 \( \text{lb/ft}^3 \).

\[ \delta \text{ (density)} \]

\( \text{(this is a force per unit volume)} \)

\[ y = 15 \]

\[ y = \frac{5}{2} x, \quad x = \frac{2}{5} y \]

\[ (\frac{4}{5} y, y) \]

\[ \Delta y \]

\[ \Delta V \approx \]

\[ \Delta F \approx \]

\[ \Delta W \approx \]

So,

\[ W = \int_0^{10} (15-y) \pi \delta (\frac{3}{2}y)^2 \, dy \]

\[ = \pi \delta \left( \frac{3}{8} \right) \int_0^{10} 15y^2 - y^3 \, dy \]

\[ = \frac{16\pi}{3} \left[ 5y^3 - \frac{y^4}{4} \right]_0^{10} \]

\[ = \frac{16\pi}{3} \left[ 10^3 \left( 5 - \frac{25}{4} \right) \right] \]

\[ = \frac{16\pi}{3} \left[ \frac{10^3 \cdot 7.5}{2500} \right] \]

\[ \approx 78,400 \text{ foot-pounds} \]