Math 1210-2
Monday Dec. 3

- We need to finish discussing curve lengths and areas of surfaces, from Friday.
- Then, §5.5 Work:

In physics, the work W done to move an object a distance d against a force F is

$$W = F \cdot d$$

example to lift a mass m a height h takes $W = (mg)\,h$ work

force of gravity on object.

units: in metric system

F units: $1\ kg\ m/sec^2 = 1$ Newton
W units: 1 Newton-meter = 1 joule

in English system
F units: pounds (1 pound $= 1$ slug ft/sec^2)
W units: 1 foot-pound

in a closed physical system, work converts Kinetic energy (KE) to potential energy (PE), and the total energy := KE + PE is constant.

example For a thrown object (neglecting friction),

$$KE + PE = \tfrac{1}{2}mv^2 + mgh \quad \text{is constant}$$

check: take $\frac{d}{dt}$ (total energy)

$$= mvv'(t) + mgv$$
$$= mv\left(v'(t) + g\right)$$
$$= 0 \text{ iff } h''(t) = -g$$

Newton's Law.

forces can change depending on location
e.g. Earth's gravitational attraction on an object is really

$$|F| = \frac{GMm}{r^2}$$

G = univ const
M = mass of earth
m = object mass
r = distance to center of Earth

so in fact $\frac{GM}{R^2} = g = 9.8\ m/sec^2$

for R = radius of Earth

so,

$$|F| = mg\left(\frac{R^2}{r^2}\right)$$

Work computation for space-varying force:

$F = F(x)$

work done to move object from a to b

$$= W \approx \sum_{i=1}^{n} F(\bar{x}_i)\,\Delta x_i \qquad !$$

so $$\boxed{W = \int_a^b F(x)\,dx}$$

<u>Exercise 1</u>: How much work must be done on an object to move it from the surface of the Earth to ∞, neglecting friction?

$$W = \int_R^\infty \underbrace{mg\left(\frac{R^2}{r^2}\right)}_{\text{Force}} \underbrace{dr}_{\text{dist}} =$$

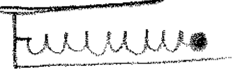
<u>Exercise 2</u>:
What should the initial speed of object be to barely escape the Earth?

Use $KE + PE = \text{const}$ to recompute escape velocity $v_0 = \sqrt{2gR} \approx 6.93$ miles/sec

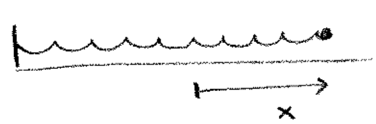
(want $v \to 0$ as $r \to \infty$, so at ∞ $KE = 0$)

for contrast, see the derivation using Newton's Law & separable DE's on page 207, and Nov. 5 lecture notes

SPRINGS



spring at rest



stretched spring
(compressed if $x < 0$)

Hooke's Law $F(x) = kx$ $\qquad k$ = spring constant

is really just the tangent line approximation ("linear approx") to a differentiable force function which satisfies $F(0) = 0$.

Exercise 3 (p 302) If natural length of a spring is .2 m and 12 N force is required to stretch it an additional .04 m

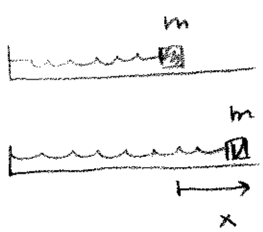
(1) find the spring constant k

(2) how much work is done in stretching the spring from its natural length to .3 m?

spring dynamics [and springs are everywhere in Science]

set total energy = 0 for mass-spring at rest.

$$W = \int_0^x ks\,ds = \tfrac{1}{2}ks^2\Big]_0^x = \tfrac{1}{2}kx^2$$



$$E = \underbrace{\tfrac{1}{2}kx^2}_{PE} + \underbrace{\tfrac{1}{2}mv^2}_{KE}$$

$x'(t) = v$

$$\frac{dE}{dt} = 0 = k x x'(t) + m x'(t) x''(t)$$
$$= x'(t)\left[kx + m x''(t)\right]$$

deduce $x''(t) = -\frac{k}{m} x(t)$

[could get this directly from Newton too]

solution to

$$x''(t) = -\omega_0^2 x(t)$$

is

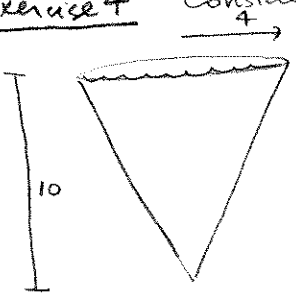
$$x(t) = C\cos(\omega_0 t - \phi)$$
$$= A\cos\omega_0 t + B\sin\omega_0 t$$

and this is the primary reason for the importance of trigonometry is science

(p. 303) (A different kind of work problem)

Exercise 4 Consider a conical water tank, height = 10 feet

$r = 4$ feet



full of water.

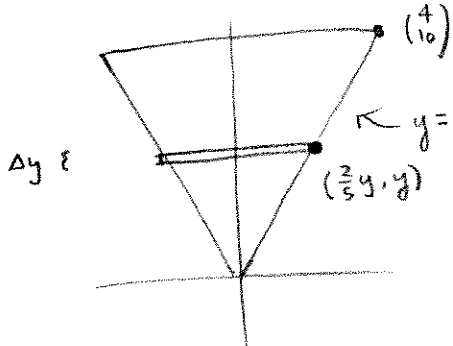
How much work must be done to pump all of the water in the tank to a height 5 feet above the top of the tank?

Water weighs 62.4 lb/ft^3

// δ (density)

(this is a force per unit volume)

$y = 15$



vol $\quad \Delta V \approx$

force $\quad \Delta F \approx$

work $\quad \Delta W \approx$

So, $$W = \int_0^{10} (15-y)\,\pi\delta\,(.4y)^2\,dy$$

$$= \pi\delta(.16)\int_0^{10} 15y^2 - y^3\,dy$$

$$= .16\pi\delta\left[5y^3 - \frac{y^4}{4}\right]_0^{10}$$

$$= .16\pi\delta\left[\underbrace{10^3(5-2.5)}_{2500}\right]$$

$$\approx 78,400 \qquad \text{foot-pounds}$$