Math 1210-2
Tuesday Aug 27

§P.5 Definite integrals : an important application of antidifferentiation

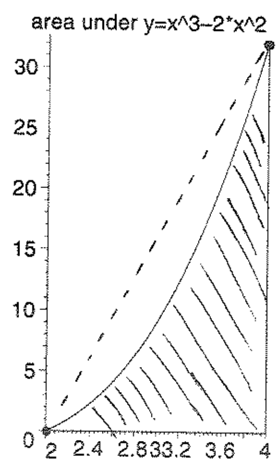
Area under graphs :

Definition A: A function $f(x)$ is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.
(In other words, as x approaches x_0 the values of $f(x)$ approach $f(x_0)$)

If $f(x)$ is continuous at every x_0 in the interval $[a,b]$, then f is continuous on the interval. (This makes f "good")

If f is continuous on the interval $[a,b]$, and if $f(x)$ is ≥ 0 for each x in the interval, there is a well-defined area bounded on the left by $x=a$, on the right by $x=b$, above by $y=f(x)$, below by the x-axis.
We call this area under the curve $y=f(x)$, between $x=a,b$, and write

$$\boxed{\int_a^b f(x)\,dx}$$ "the (definite) integral of $f(x)$, from $x=a$ to $x=b$"



Example 1 : $\int_2^4 x^3 - 2x^2\,dx$ stands for the shaded area here:

Exercise 1 : Use the indicated Δ to estimate the shaded area

Amazing fact, called Fundamental Theorem of Calculus (§4.3-4.4):
Theorem A : If f is a non-negative continuous function on the interval $[a,b]$, and F is any antiderivative of f, then

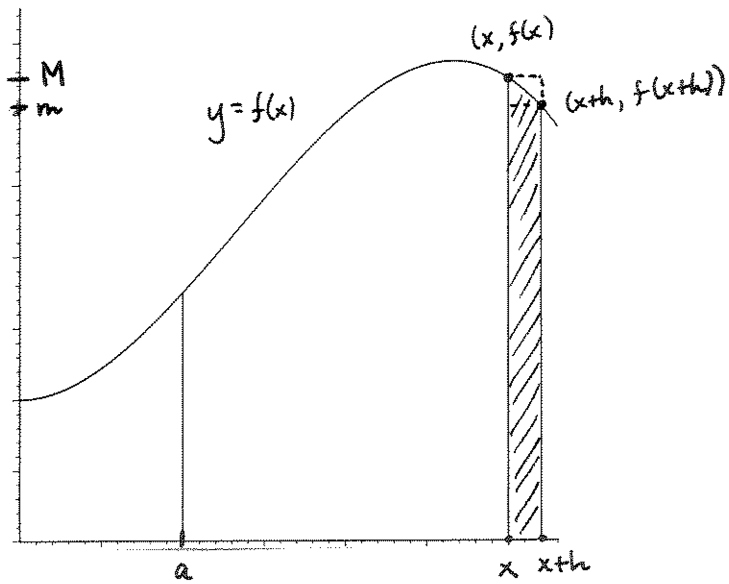
$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$ often write $F(x)\Big]_a^b$ for $F(b)-F(a)$

Exercise 2 : Find the exact area you estimated in previous exercise

Why Theorem A is true (rough idea):

Let $A(x)$ be the area under the graph of f, between a and x:

Then $A(x+h) - A(x)$ is the area of the shaded region (for $h>0$, small):



$$mh \leq \text{shaded area} \leq Mh$$

where m is the minimum value of f between x & $x+h$
M " " maximum " " " " "

So, (divide by h, $h>0$),

$$m \leq \frac{A(x+h) - A(x)}{h} \leq M$$

As $h \to 0$, m, M both approach $f(x)$, because f is continuous

Conclude
$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$
$$\text{''}$$
$$A'(x).$$

Thus the area function is an antiderivative of $f(x)$.

If $F(x)$ is any other antiderivative, then from Monday notes,

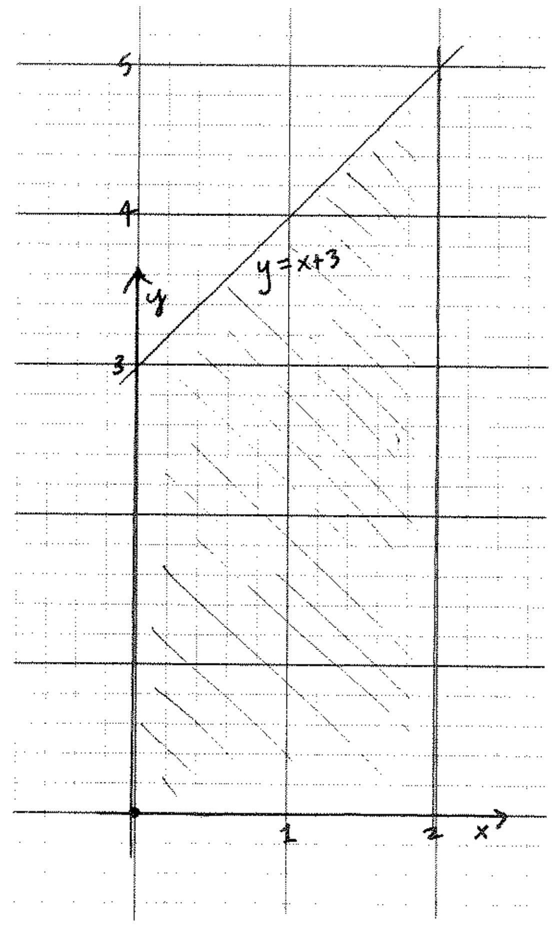
$$A(x) = F(x) + C$$

Since $A(a) = 0 = F(a) + C$, deduce $C = -F(a)$

$$A(x) = F(x) - F(a)$$

$$\int_a^b f(x)\,dx = A(b) = F(b) - F(a) \quad !$$

Exercise 3 Use geometry and the FTC to compute the area under $y = x+3$, between $x=0,2$



geometry:

FTC:

Exercise 4: $\int_0^1 x^2\,dx =$

$\int_0^1 x^3\,dx =$

$\int_0^1 x^n\,dx =$

The reason for the (strange) notation:

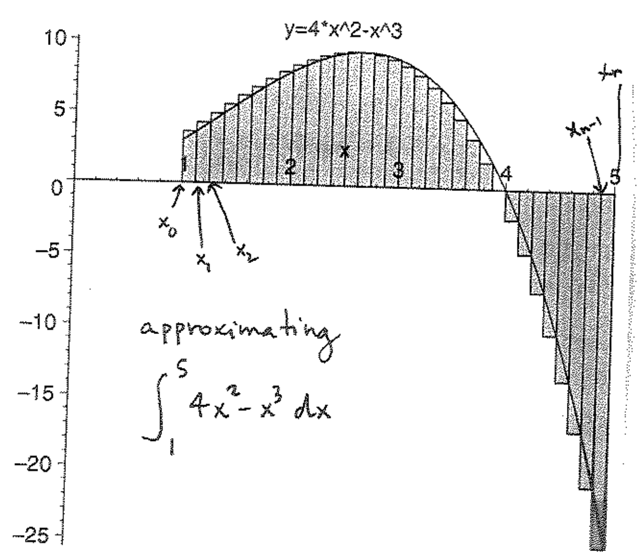
When $f(x)$ is not necessarily positive,

$$\int_a^b f(x)\,dx$$

represents the <u>signed area</u> from $x=a$ to b, between $y=f(x)$ and the x-axis, i.e. the sum of the areas above the x-axis, <u>minus</u> the areas below. This can be approximated by subdividing $[a,b]$ into n subintervals of width $h = \frac{b-a}{n}$, and computing

$$f(x_1)\underbrace{(x_1-x_0)}_{h} + f(x_2)\underbrace{(x_2-x_1)}_{h} + \dots + f(x_n)\underbrace{(x_n-x_{n-1})}_{h}$$

$$= \sum_{i=1}^{n} f(x_i)(x_i-x_{i-1}).$$



$\int_a^b f(x)\,dx$ is the limit as $h \to 0$!

$\int$ stands for sum!

<u>AND</u> FTC still holds,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Physics interpretation: (of FTC)

$$\int_a^b v(t)\,dt = s(b) - s(a) \qquad \text{(net displacement)}$$

and can be computed with any antiderivative of the velocity function

<u>Exercise 5</u> Suppose a particle moves along a straight (number) line, horizontal, positive direction to the right. with velocity at time t

$$v(t) = t^2 - t^3 \qquad \text{meters/minute}$$

How far to the left or the right of the initial position (@ $t=0$) will the particle be after 2 minutes?

tomorrow (Wednesday) we begin Varberg @ Chapter 0.4
You may wish to review 0.1–0.3