5P.4 Antiderivatives of functions
(also called antidifferentiation)

**Definition A** If \( f(x) \) is a function, then an antiderivative for \( f \) is a function having \( f(x) \) as its derivative.
(This is the reverse procedure to taking derivatives.)

**Exercise 1:** Using your knowledge of derivatives and working backwards, find all the antiderivatives of \( f(x) = 3x \) that you can find. The pictures to the right are a hint.

- **Definition B** If \( f(x) \) is a function, the set of antiderivatives for \( f(x) \) is denoted
  \[ \int f(x) \, dx \]
  and is called the **indefinite integral of** \( f(x) \).  
  (with respect to \( x \))

**Exercise 2**
\[ \int 3x \, dx = \]

Note: the "\( x \)" in "\( dx \)" says that \( x \) is your variable.

(you will rely on the theorem at the top of the next page to answer this exercise completely)
Theorem (We'll prove this in §3.8 of Vanberg, but it's believable.)

If \( F(x) \) is an antiderivative of \( f(x) \), then all other antiderivatives are of the form \( F(x) + C \), where \( C \) is a constant.

i.e. \( \int f(x) \, dx = F(x) + C \)

Exercise 3: Work backwards, as in Exercise 1, to find

3a) \( \int x^3 \, dx \)

3b) \( \int 4 \, dx \)

3c) \( \int x^3 + 4 \, dx \)

3d) \( \int 2x^5 + 3x^3 + x - 10 \, dx \)

3e) \( \int -4t + 2 \, dt \)

3f) \( \int t^2 x^3 \, dt \)

\( \int t^2 x^3 \, dx \)
Hopefully on page 2 we discovered

**Theorem A** If \( n \) is a non-negative integer, then
\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C
\]
(where \( C \) ranges over all constants)

\[\text{e.g.}\]
\[
\int 1 \, dx = x + C \\
\int x \, dx = \frac{x^2}{2} + C \\
\int x^2 \, dx = \frac{x^3}{3} + C \\
\int x^3 \, dx = \frac{x^4}{4} + C \\
etc.
\]

**proof of Theorem A:** Let \( F(x) = \frac{x^{n+1}}{n+1} \). Then \( F'(x) = \frac{1}{n+1} (n+1) x^n = x^n \).

This shows \( F(x) \) is an antiderivative of \( x^n \). You get every other antiderivative by adding (arbitrary) constants to \( F(x) \).

**Theorem B** If \( f \) and \( g \) are functions and \( "a" \) is a constant, then
\[
(1) \int a f(x) \, dx = a \int f(x) \, dx \\
(2) \int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx
\]

**proof:** If \( F(x) \) is an antiderivative of \( f(x) \), i.e. \( F'(x) = f(x) \), then \( (aF)' = a F' = af \)

so \( aF(x) \) is an antiderivative of \( af(x) \).

Thus
\[
\int a f(x) \, dx = a \, F(x) + C = a \left( F(x) + C \right)
\]
\[\qquad (C = C_1/a)\]

Similarly, if \( F' = f \) and \( G' = g \), then \( (F+G)' = f + g \)

so
\[
\int f(x) + g(x) \, dx = F(x) + G(x) + C
\]
\[= \int f(x) \, dx + \int g(x) \, dx\]
Exercise 4: Find the antiderivative of $x^2 - 1$ which has value 2 when $x = -2$.

Exercise 5: The acceleration due to gravity on the earth's surface is $g = -32 \text{ ft/sec}^2$ ($= -9.8 \text{ m/sec}^2$).

A ball is thrown straight upward with initial velocity of $128 \text{ ft/sec}$ after which only gravity accelerates it. (We neglect friction.)

5a) What is the velocity $v(t)$, if $t = 0$ sec, is when object is released?

5b) When does object reach maximum height?

5c) If ball was thrown from initial height of 6 feet, how high does it go?

5d) When does it hit the ground?