Variations on the Projective Central Limit Theorem

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A project submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of
Master of Statistics Emphasis Mathematics

Department of Mathematics
University of Utah
2010
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1 Introduction

The projective central limit theorem asserts the following.

**Theorem 1.1.** Let $U^n = (U^n_1, U^n_2, ..., U^n_n)$ be a sequence of uniform random variables on centered $n$-spheres of radius $\sqrt{n}$. As the dimension $n$ tends to infinity, the first $k$ coordinates $(U^n_1, ..., U^n_k)$ converge in distribution to a $k$-variate standard normal:

$$(U^n_1, ..., U^n_k) \to_d N(0, I_k)$$

This paper will introduce the following related results.

**Theorem 1.2.** Let $U^n = (U^n_1, U^n_2, ..., U^n_n)$ be a sequence of uniform random variables on centered $n$-balls of radius $\sqrt{n}$. For $a < b$ as $n \to \infty$

$$P(a \leq U^n_1 \leq b) \to P(a \leq N(0, 1) \leq b).$$

**Theorem 1.3.** Let $U^n$ be a sequence of uniform random variables on centered unit $n$-cubes. With $\pi U^n$ denoting the projection of $U^n$ onto the diagonal spanned by $(1, 1, ..., 1)$ and $T^n_c = \|\pi U^n\|_2$ then as $n \to \infty$

$$T^n_c \to_d N(0, 1/12).$$

**Theorem 1.4.** Let $U^n$ be a sequence of uniform random variables on the boundaries of centered unit $n$-cubes. With $\pi U^n$ denoting the projection of $U^n$ onto the diagonal spanned by $(1, 1, ..., 1)$ and $T^n_b = \|\pi U^n\|_2$ then as $n \to \infty$

$$T^n_b \to_d N(0, 1/12).$$

Evidence for Theorem 1.4 was initially obtained through statistical simulation. An analytic proof was later found using the Lyapunov Central Limit Theorem. We will present both approaches in later sections.

In order to prove the above-mentioned theorems we will have to define precisely what is meant when we state that a random variable is distributed uniformly on a body or surface in $\mathbb{R}^n$. For bodies we can use top-dimensional Lebesgue measure. For surfaces we will use a suitable parametric definition of area. We will have the need to speak of the $(n-1)$-volume of $(n-1)$-dimensional sets and the technical development in Section 2 will introduce the appropriate definitions and proceed to prove geometric propositions regarding integration. For example:

**Proposition 1.5.** (Fubini's Theorem for Sets) Let $E$ be a measurable subset of $\mathbb{R}^n$ and $E_{x_n} = \{(x_1, x_2, ..., \hat{x}_n) \in \mathbb{R}^{n-1} : (x_1, x_2, ..., x_n) \in E\}$. With $L^k$ denoting $k$-dimensional Lebesgue measure

$$\int_E dL^n = \int_{-\infty}^{\infty} L^{n-1}(E_{x_n}) dL^1(x_n).$$

Such propositions provide the support needed to rigorously prove Theorem 1.2, Theorem 1.3, and Theorem 1.4. These proofs are provided in later sections.
2 Technical development

The results of this preparatory section are interesting in their own right. Highlights include multiple versions of Fubini’s Theorem, formulas for the volumes of balls in various dimensions, formulas for the surface areas of spheres in various dimensions, and also Sterling’s approximation for the Gamma function.

We assume here at the outset that all encountered sets and functions are measurable.

Definition 2.1. The n-dimensional, Lebesgue measure of a set $V \subseteq \mathbb{R}^n$ is defined to be

$$L^n(V) = \inf \left\{ \sum_{i=1}^{\infty} \prod_{j=1}^{n} (b_{ij} - a_{ij}) : V \subseteq \bigcup_{i=1}^{\infty} \prod_{j=1}^{n} (a_{ij}, b_{ij}) \right\},$$

where for fixed $i$, $a_{ij} < b_{ij}$ for every $j$.

For a fixed $i$, we call $\prod_{j=1}^{n} (a_{ij}, b_{ij})$ an n-dimensional box. The inf is taken over all countable covers of $V$ by n-dimensional boxes.

By using $L^1$ we can define a product measure on $\mathbb{R}^n$.

Definition 2.2. If we let each $D_{ij}$ be a measurable subset of $\mathbb{R}$ then we define the n-dimensional, Lebesgue, product measure of a set $V \subseteq \mathbb{R}^n$ to be

$$L^1 \times L^1 \times \ldots \times L^1(V) = \inf \left\{ \sum_{i=1}^{\infty} \prod_{j=1}^{n} L^1(D_{ij}) : V \subseteq \bigcup_{i=1}^{\infty} \prod_{j=1}^{n} D_{ij} \right\}.$$

This time we cover with ‘boxes’ that are cartesian products of measurable sets. They are not necessarily products of intervals. We can state the following though.

Theorem 2.3. For every set $V \in \mathbb{R}^n$

$$L^n(V) = L^1 \times L^1 \times \ldots \times L^1(V).$$

This theorem is a consequence of the well known Caratheodory Extension Theorem (see Khoshnevisan [3]) which we state in a simplified and weaker form. In the statement, a box is a product of intervals.

Theorem 2.4. If two measures agree on the set of all finite unions of boxes, then they agree on the $\sigma$-algebra generated by the set of all finite unions of boxes.

This will prove useful in our treatment of uniform random variables.

In short, Lebesgue measure is used to define integration (see [4]).

Definition 2.5. A function is integrable over $E \subseteq \mathbb{R}^n$ if

$$\int_E |f| dL^n < \infty.$$
**Theorem 2.6.** (Dominated Convergence Theorem) Let \( \{f_n\} \) be a sequence of functions converging pointwise to \( f \) on some set \( E \). If on \( E \) there exists an integrable function \( g \) such that \( |f_n| \leq g \), then as \( n \to \infty \)

\[
\int_E f_n \to \int_E f.
\]

For more discussion and a proof of Theorem 2.6 see (Cheney [4]). Theorem 2.6 will allow us to infer weak convergence of random variables based on pointwise convergence of their density functions.

Another important result that we will need states how a mapping effects Lebesgue Measure.

**Theorem 2.7.** Let \( f = (f_1(x_1, x_2, ..., x_n), f_2(x_1, x_2, ..., x_n), ..., f_n(x_1, x_2, ..., x_n)) \) be a \( C_1 \) function from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). For a measurable set \( E \subseteq \mathbb{R}^n \)

\[
L^n(f(E)) = \int_E |\text{det}J|,
\]

where \( J \) is the Jacobian matrix of partial derivatives

\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}.
\]

We will sometimes speak of \( |\text{det}J| \) as the expansion factor for the mapping \( f \). Note that its value depends on the domain variable.

Another important theorem, which we will see in many forms, is Fubini’s theorem (see Royden [5] for more details and proofs). Here we state a simplified version. Note that the hat notation signals the omission of that variable.

**Theorem 2.8.** (Fubini’s Theorem for Sets) Let \( E \) be a subset of \( \mathbb{R}^n \), and \( E_{x_n} = \{(x_1, x_2, ..., x_n) \in E^* : (x_1, x_2, ..., x_{n-1}) \in E \} \). Then

\[
\int_E dL^n = \int_{-\infty}^{\infty} L^{n-1}(E_{x_n}) dL^1(x_n).
\]

If for each \( x_n \) we identify \( \mathbb{R}^{n-1} \) with \( (\mathbb{R}^{n-1}, x_n) \) we can then interprete Theorem 1.7 as stating that the volume of a set is equal to an integral of the area (\( L^{n-1} \) measure) of horizontal slices of that set. There’s nothing special about \( x_n \) though. We don’t need the slices to be horizontal. We can scan through \( E \) in any direction. This amounts to picking a point \( e_0 \in E \) and a unit vector \( v \), resulting in the line

\[
e_0 + tv : t \in \mathbb{R}.
\]

Although points on this line are actually within \( \mathbb{R}^n \) we will speak of them as indexed by \( t \). We define the slice at \( t \) by first choosing a \( t_1 \) where \( (e_0 + t_1v) \neq 0 \) so that we can define the orthogonal slice at \( t \) to be

\[
E_t = \{e \in E : \langle e - (e_0 + tv), (e_0 + t_1v) \rangle = 0 \}.
\]
We can define the area of this slice by first rotating it so that it’s horizontal, then using the above identification of $\mathbb{R}^{n-1}$ with $(\mathbb{R}^{n-1}, x_n)$, and finally using $L^{n-1}$ to measure area. This definition of area is indeed well defined, and a special case of the parametric definition that we will give shortly. In this terminology we can state the following corollary to Theorems 2.7 and 2.8.

**Corollary 2.9.** For a Borel measurable set $E$ we have

$$\text{volume}(E) = \int_{E} 1 dL^n = \int_{-\infty}^{\infty} \text{area}(E_t) dL^1(t).$$

Corollary 2.9 says that no matter which direction we choose to travel through $E$, the integral of the area of the orthogonal slices will give the volume.

We can expand on this idea as follows. Starting at $e_0 \in E$ we can travel outwards and integrate the surface areas of concentric spherical ‘slices’ to get the volume. We will only need the case where $E$ is a ball and $e_0$ the center of the ball. We will call this result Fubini’s Theorem in Spherical Coordinates, but first we need some more development. We start by formally defining area.

**Definition 2.10.** Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n) : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be a $C^1$ function on an open set $O \subseteq \mathbb{R}^{n-1}$. Assume that $\phi$ is 1 to 1 on a measurable set $E \subseteq O$ and also that the Jacobian matrix

$$J = \begin{pmatrix}
\frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_{n-1}} \\
\frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_{n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_{n-1}}
\end{pmatrix} = (v_1 \ v_2 \ \cdots \ v_{n-1})$$

is of full rank on $E$ so that we can define a unit vector $n$ that is normal to the column space of $J$, which we use to define

$$K_\phi = (v_1 \ v_2 \ \cdots \ v_{n-1} \ n),$$

which we use to define the area of $\phi(E)$:

$$\text{area}(\phi(E)) = \int_{E} |\text{det}(K_\phi)|.$$
Theorem 2.11. If \( \phi_1 \) is \( C_1 \) and 1 to 1 on \( E_1 \) and \( \phi_2 \) is \( C_1 \) and 1 to 1 on \( E_2 \), and \( \phi_1(E_1) = \phi_2(E_2) \) then
\[
\text{area}(\phi_1(E_1)) = \text{area}(\phi_2(E_2)).
\]

Proof.

\[
\begin{align*}
\text{area}(\phi_1(E)) &= \text{area}(\phi_2\phi_2^{-1}\phi_1(E_1)) \\
&= \text{area}(\phi_2(\phi_2^{-1}\phi_1(E_1))) \\
&= \text{area}(\phi_2(E_2))
\end{align*}
\]

Note that we have now generalized our definition for the areas of slices. The area of a slice is defined via any parametrization of the slice by using definition 2.10.

We can even speak of the area of sets defined piecewise as a union \( \bigcup_{j=1}^m \phi_j(E_j) \) of images. As long as the image sets overlap pairwise on sets of measure zero we can define the area of the union as the sum of the areas:
\[
\text{area}\left(\bigcup_{j=1}^m \phi_j(E_j)\right) = \sum_{i=1}^m \text{area}(\phi_j(E_j)).
\]

Sometimes a set will be defined independently of a parametrization. For example, consider a sphere of radius \( r \) in \( \mathbb{R}^n \):
\[
S^n_r := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \ldots + x_n^2 = r^2\}.
\]

Even though this sphere is an \((n - 1)\)-dimensional object, we will denote it with a superscript of \( n \) because the ambient space is \( n \)-dimensional. This will simplify later notation. We use the same notational conventions for balls:
\[
B^n_r := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \ldots + x_n^2 < r^2\}.
\]

Note that we have defined an open ball. The open \((n-1)\)-ball of radius \( r \) leads to a parametrization (with two pieces) of \( S^n_r \), or actually \( S^n_r \) minus the equator, a set with zero area.

We can parametrize the top with
\[
\phi_t : B^{n-1}_r \to \mathbb{R}^n \\
\phi_t : (x_1, \ldots, x_{n-1}) \mapsto \left(x_1, \ldots, x_{n-1}, \sqrt{1 - (x_1^2 + \ldots + x_{n-1}^2)}\right)
\]

and the bottom with
\[
\phi_b : B^{n-1}_r \to \mathbb{R}^n \\
\phi_b : (x_1, \ldots, x_{n-1}) \mapsto \left(x_1, \ldots, x_{n-1}, -\sqrt{1 - (x_1^2 + \ldots + x_{n-1}^2)}\right).
\]
so that
\[ \text{area}(S^n_r) = \int_{B^n_{r-1}} |\det(K_{\phi_1})| + \int_{B^n_{r-1}} |\det(K_{\phi_2})|. \]
Even though this simplifies to
\[ \text{area}(S^n_r) = 2 \int_{B^n_{r-1}} |\det(K_{\phi_1})|, \]
this is still just a definition for the area. The integral might not be so easy to compute. Although, we can state it in terms of the gamma function
\[ \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dL_1(t). \]
For this we need a different parametrization and some more theory.

**Theorem 2.12.** (Fubini’s Theorem for Functions) Let \( E \) be a measurable set in \( \mathbb{R}^n \) and \( f: \mathbb{R}^n \to \mathbb{R} \) be a continuous function. Then
\[
\int_E f dL^n = \int_{-\infty}^{\infty} \left( \int_{E_{x_n}} f L^{n-1}(x_1, ..., x_{n-1}, x_n) \right) dL^1(x_n).
\]
Compare this with our earlier Fubini’s Theorem for Sets (Theorem 2.8). Theorem 2.12 now allows for a function \( f \), which may be an expansion factor (see Theorem 2.7). For a proof of Theorem 2.12 see Royden [5].

Consider the expansion factor that arises when putting spherical coordinates on \( \mathbb{R}^n \), or actually \( \mathbb{R}^n \) minus a set of \( L^n \) measure zero. Letting \( C = (0, \pi) \times ... \times (0, \pi) \times (0, 2\pi) \) we have
\[
\psi: \mathbb{C} \times \mathbb{R}^+ \subseteq \mathbb{R}^n \to \mathbb{R}^n
\]
and we can think of these coordinates as parametrizing (for each \( r \)) the sphere \( S^n_r \), or at least \( S^n_r \) minus a set of zero area. Since \( |\frac{\partial \psi}{\partial r}| \equiv 1 \) we can identify \( \det J \psi \) with \( \det K_{\psi_r} \) where \( \psi_r \) maps \( \mathbb{R}^{n-1} \sim (r, \mathbb{R}^{n-1}) \) into and almost onto \( S^n_r \). So our definition of area and Fubini’s theorem for functions results in
\[
\text{volume}(B^n_R) = \int_{B^n_{R-1}} 1 dL^n
\]
\[
= \int_{C \times (0, R)} |\det J \psi| dL^n
\]
\[
= \int_0^R \left[ \int_{C_r} |\det J \psi| dL^{n-1}(\theta_1, ..., \theta_{n-1}) \right] dL^1(r)
\]
\[
= \int_0^R \text{area}(S^n_r) dL^1(r)
\]
which we summarize in the following corollary.
Corollary 2.13. (Fubini’s Theorem in Spherical Coordinates)

\[
\text{volume}(B^n_R) = \int_0^R \text{area}(S^n_r) dL^1(r).
\]

This result allows us to integrate surface areas of spheres to get volumes of balls, but we still do not have a formula for area\((S^n_r)\). To this aim observe the following.

How does \(\text{volume}(B^n_R)\) relate to \(\text{volume}(B^n_1)\)? It is \(r^n\) times as large. This is due to the definition of Lebesgue measure.

How does \(\text{area}(S^n_r)\) relate to \(\text{area}(S^n_1)\)? It is \(r^{n-1}\) times as large. This is also a consequence of Lebesgue measure, and our parametric definition of area.

So we have the following:

\[
\text{volume}(B^n_r) = r^n \text{volume}(B^n_1)
\]

and

\[
\text{area}(S^n_r) = r^{n-1} \text{area}(S^n_1).
\]

Combining this with Corollary 2.13 results in

\[
\text{volume}(B^n_1) = \int_0^1 \text{area}(S^n_r) dL^1(r)
= \int_0^1 r^{n-1} \text{area}(S^n_1) dL^1(r)
= \text{area}(S^n_1)/n.
\]

Thus for \(n\) dimensional unit balls, the surface area is always \(n\) times the volume.

We can check this formula in dimensions two and three:

\[
\text{volume}(B^2_1) = \pi = 2\pi/2 = \text{area}(S^2_1)/2,
\]

\[
\text{volume}(B^3_1) = \frac{4}{3}\pi = 4\pi/3 = \text{area}(S^3_1)/3.
\]

So we have agreement with the well known lower-dimensional formulas.

Notice that equation 2.1 tells us that if we could only derive a formula for \(\text{area}(S^n_1)\), then we would have a formula for \(\text{volume}(B^n_1)\) as well. To obtain this formula observe that the integrand of the third expression in (2.1) resembles the integrand of the gamma function. After adding in an exponential term, and increasing the upper bound to \(\infty\); we arrive at the following, due to a change of coordinates and multiple uses of Fubini’s theorem for functions:

\[
\int_0^\infty \text{area}(S^n_1)r^{n-1}e^{-r^2} dL^1(r)
= \int_{-\infty}^\infty ... \int_{-\infty}^\infty e^{-(x_1^2 + ... + x_n^2)} dL^1(x_1) ... dL^1(x_n)
= \left(\int_{-\infty}^\infty e^{-x^2} dL^1(x)\right)^n
= \pi^{n/2}.
\]
The last equality holds since $\text{area}(S^2_r) = 2\pi r$. This results in:

$$
\int_{-\infty}^{\infty} e^{-x^2} dL^1(x) = \left( \int_{-\infty}^{\infty} e^{-x^2} dL^1(x) \int_{-\infty}^{\infty} e^{-y^2} dL^1(y) \right)^{\frac{1}{2}}
$$

$$
= \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} dL^1(x) e^{-y^2} dL^1(y) \right)^{\frac{1}{2}}
$$

$$
= \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dL^1(x) dL^1(y) \right)^{\frac{1}{2}}
$$

$$
= \left( \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dL^2(x,y) \right)^{\frac{1}{2}} = \left( \int_0^{\infty} 2\pi re^{-r^2} dL^1(r) \right)^{\frac{1}{2}} = \pi^{1/2}.
$$

On the other hand, with the substitution $u = r^2$ we have

$$
\int_0^{\infty} \text{area}(S^n_1) r^{n-1} e^{-r^2} dr = \text{area}(S^n_1) \frac{1}{2} \int_0^{\infty} u^{\frac{n}{2}-1} e^{-u} du = \text{area}(S^n_1) \frac{1}{2} \Gamma(n/2), \quad (2.3)
$$

where the last equality is the definition of the gamma function.

Combining equations (2.2) and (2.3) then results in

$$
\text{area}(S^n_1) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad (2.4)
$$

and by equation (2.1)

$$
\text{volume}(B^n_1) = \frac{2\pi^{n/2}}{n\Gamma(n/2)}. \quad (2.5)
$$

The utility of expressing these formulas in terms of the gamma function arises because the gamma function is well known (see [8] for a catalog of formulas). Of particular interest to us is the following.

**Theorem 2.14.** (Stirling’s Formula)

$$
\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left( \frac{x}{e} \right)^x \left( 1 + O \left( \frac{1}{x} \right) \right). \quad (2.6)
$$

This is the last result of this preparatory section. We have defined volume and area in $\mathbb{R}^n$ as well as derived formulas for the volume of balls and surface area of spheres. Our essential tool was Lebesgue measure and we used it to show that in many instances the integral of area gives a volume. We now move onto probability theory.
3 Uniform random variables on n-balls and n-spheres

We begin by presenting a slight variant of the standard proof of Theorem 1.1. Then we prove Theorem 1.2 and see that our technique can be used to construct an alternative proof of Theorem 1.1, in the case of projecting onto a single coordinate.

3.1 Standard proof of Theorem 1.1

We begin with a definition.

**Definition 3.1.** A random variable \( U \) is uniformly distributed on the sphere \( S^n_r \) if for all (measurable) \( E \subseteq S^n_r \) we have

\[ P(U \in E) = \frac{\text{area}(E)}{\text{area}(S^n_r)}. \]

In order to prove Theorem 1.1 we need to show that if

\[ U^n = (U^n_1, U^n_2, \ldots, U^n_n) \]

is a sequence of uniform random variables on centered \( n \)-spheres of radius \( \sqrt{n} \), then as \( n \to \infty \)

\[ (U^n_1, U^n_2, \ldots, U^n_k) \to^d N(0, I_n). \]

We appeal to the strong law of large numbers.

**Theorem 3.2.** (Strong Law of Large Numbers) Let \( X_1, X_2, \ldots \) be independent, identically distributed random variables with \( E(X_1) < \infty \). Then

\[ \lim_{n \to \infty} \frac{\sum_{i=1}^{n} X_i}{n} = E(X_1) \text{ a.s.} \]

**Proof.** see Khoshnevisan [3].

Note that convergence a.s. implies convergence in distribution. Next we state and prove a lemma.

**Lemma 3.3.** Let \( Z^n \) be standard, \( n \)-variate, normal random variable. Then

\[ \frac{\sqrt{n}}{\|Z\|} Z \]

is uniform on the \( n \)-sphere of radius \( \sqrt{n} \)

**Proof.** Note that the factor \( \frac{\sqrt{n}}{\|Z\|} \) projects \( Z^n \) onto \( S^n_{\sqrt{n}} \). In spherical coordinates this is given by

\[ (R, \Theta_1, \ldots, \Theta_{n-1}) \mapsto (\sqrt{n}, \Theta_1, \ldots, \Theta_{n-1}), \]

and the resulting density is given by

\[ \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-r^2/2}. \]
By Fubini’s theorem we then have for a (measurable) set $E \subseteq S^n_{\sqrt{n}}$

\[
P \left( \left| \frac{\sqrt{n}}{Z} \right| Z \in E \right) = P \left( \frac{Z}{|Z|} \in \frac{E}{\sqrt{n}} \right) = \int_0^\infty r^{n-1} \text{area} \left( \frac{E}{\sqrt{n}} \right) \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{r^2}{2}} dL^1(r)
\]

\[
= \frac{\text{area} \left( \frac{E}{\sqrt{n}} \right)}{\text{area} \left( S^n_{\sqrt{n}} \right)} \int_0^\infty r^{n-1} \text{area} \left( S^n_{1} \right) \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{r^2}{2}} dL^1(r)
\]

\[
= \frac{\text{area} \left( \frac{E}{\sqrt{n}} \right)}{\text{area} \left( S^n_{\sqrt{n}} \right)}
\]

\[
= \frac{\text{area}(E)}{\text{area}(S^n_{\sqrt{n}})}.
\]

\[\square\]

We can now prove the theorem. We let $Z^n$ denote an $n$-variate standard normal and $N$ denote a single-variate standard normal. By lemma 3.1

\[
U^n \sim_d \frac{\sqrt{n}}{|Z|} Z^n
\]

so the first $k$ coordinates $(U^n_1, ..., U^n_k)$ of $U^n$ satisfy

\[
(U^n_1, ..., U^n_k) = \frac{\sqrt{n}}{|Z|} Z^k.
\]

Since

\[
\frac{|Z|}{\sqrt{n}} = \frac{\sqrt{N^2 + ... + N^2}}{\sqrt{n}} \quad \text{and} \quad E(N^2) = 1
\]

the strong law of large numbers then shows that

\[
\frac{|Z|}{\sqrt{n}} \to 1 \text{ a.s.}
\]

so that

\[
(U^n_1, ..., U^n_k) \to_d Z^k.
\]

Note the curious fact that made this all work: for large $n$ the magnitude of the $n$-variate standard normal is close to $\sqrt{n}$ with high probability. This sheds light on how $a\pi^{-n/2}e^{-r^2/2}r^{n-1}$ behaves.

### 3.2 Proof of Theorem 1.2

Here we present a proof of Theorem 1.2. In order to prove Theorem 1.2 we need to show that if

\[
U^n = (U^n_1, U^n_2, ..., U^n_n)
\]
is a sequence of uniform random variables on centered \( n \)-balls of radius \( \sqrt{n} \), then for \( a < b \) as \( n \to \infty \)

\[
P(a \leq U^n_1 \leq b) \to P(a \leq N(0, 1) \leq b).
\]

With \( I \) denoting the indicator function the following gives the density for \( U^n \):

\[
f_{U^n}(x) = \frac{1}{\text{volume}(B^n_{\sqrt{n}})} I\{B^n_{\sqrt{n}}\}.
\]

If we project onto the first coordinate, which we relabel with \( t \), and use Fubini’s theorem, then the density will become

\[
f_{\pi U^n}(t) = \frac{1}{\text{volume}(B^n_{\sqrt{n}}) \text{volume}(B^{n-1}_{\sqrt{n-t^2}})} I\{B^1_{\sqrt{n}}\}.
\]

For \( t \) in \((-\sqrt{n}, \sqrt{n})\) we then have

\[
f_{\pi U^n}(t) = \frac{n\Gamma(n/2)}{\sqrt{n}^{n/2} (n-1)^{n/2}} \frac{1}{(n-1)\Gamma((n-1)/2)}
\]

Next we state a lemma.

**Lemma 3.4.** As \( n \to \infty \)

\[
\frac{\Gamma(n/2)}{\Gamma(n/2 - 1)} \frac{1}{\sqrt{n-1}} \to 1\sqrt{2}.
\]

**Proof.** Using Stirling’s approximation we have

\[
\frac{\Gamma(n/2)}{\Gamma(n/2 - 1)} \frac{1}{\sqrt{n-1}} = \frac{\sqrt{4\pi n} (n/2)^{n/2} (1 + O(1/n))}{\sqrt{4\pi (n-1)(n/2)^{n/2} (1 + O(2/n))} \sqrt{n-1}}
\]

\[
= \frac{1}{\sqrt{2}} e^{-\frac{1}{2}} \left( \frac{n}{n-1} \right)^{\frac{n}{2}} \frac{1}{(1 + O(2/n))} \to 1\sqrt{2}.
\]

\[\square\]

The last step can be seen by substituting \( m = \frac{n-1}{2} \). With this result in hand we can now simplify further.

\[
f_{\pi U^n}(t) = \frac{n\Gamma(n/2)}{\sqrt{n}^{n/2} 2\pi^{n/2} (n-1)\Gamma((n-1)/2)}
\]

\[
= \frac{\Gamma(n/2)}{\Gamma(n/2 - 1)} \frac{1}{\sqrt{n-1}} \frac{1}{\sqrt{n-1} \sqrt{n-1}} \frac{1}{\sqrt{2\pi} n^{n/2}} \to 1\sqrt{2} e^{-\frac{t^2}{2}} = \phi(t).
\]

Again, the last step can be seen by substituting \( m = \frac{n-1}{2} \).

So we see that the densities for the projected random variables converge pointwise to a standard Gaussian.
To see that this implies convergence in distribution of the projected random variables we will use Lebesgue’s Dominated Convergence Theorem (Theorem 2.6). We can apply the theorem because
\[ \forall n, f_{\pi U_n}(t) \leq f_{\pi U_n}(0) \to \phi(0) = \frac{1}{2\pi} \]
so
\[ \exists M < \infty : \forall n, \forall t, |f_{\pi U_n}(t)| < M. \]
This all results in the following conclusion. As \( n \to \infty \)
\[ P(a \leq \pi U^n \leq b) = \int_a^b f_{\pi U^n}(t)dL^1(t) \to \int_a^b \phi(t)dL^1(t) = P(a \leq N(0,1) \leq b). \]

### 3.3 Alternative proof of Theorem 1.1

Here we present an alternative proof of Theorem 1.1, but only in the special case of a single coordinate. We will use mostly the same techniques that were used to prove Theorem 1.2 in Section 3.2. We have defined what it means for a random variable to be distributed uniformly on a sphere (Definition 3.1). In order to prove our single-coordinate version of Theorem 1.1 we need to show that if
\[ U^n = (U_1^n, U_2^n, ..., U_n^n) \]
is a sequence of uniform random variables on centered \( n \)-spheres of radius \( \sqrt{n} \), then as \( n \to \infty \)
\[ U_1^n \to^d N(0,1). \]
The density function for \( U^n \) is given by
\[ f_{U^n}(x) = \frac{1}{\text{area}(S_{\sqrt{n}})}. \]
We plan to project each \( U^n \) onto the first coordinate, which we again will label with \( t \). How does projection change the density? It’s not quite as simple as in our previous case with the ball. This time the sphere’s surface is tilted.
Due to the tilting, the projected variable will be more dense, by a factor of \( \sqrt{\frac{n}{n-t^2}} \). By Fubini’s theorem, the formula for the density of the projection is then

\[
f_{\pi U^n}(t) = \frac{2\pi \frac{n-1}{2}}{\Gamma(\frac{n-1}{2})} (n-t^2)^{\frac{n-2}{2}} \frac{\Gamma(\frac{n}{2})}{2\pi \frac{n}{2}} \left( \frac{n}{n-t^2} \right)^{\frac{1}{2}} I(|t| \leq \sqrt{n}).
\]

We again appeal to (Lemma 3.4) while taking the limit. As \( n \to \infty \)

\[
f_{\pi U^n}(t) = \frac{2\pi \frac{n-1}{2}}{\Gamma(\frac{n-1}{2})} (n-t^2)^{\frac{n-2}{2}} \frac{\Gamma(\frac{n}{2})}{2\pi \frac{n}{2}} \left( \frac{n}{n-t^2} \right)^{\frac{1}{2}}
\]

\[
= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \frac{\sqrt{n-1}}{\sqrt{n-1}} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \left( \frac{n-t^2}{n} \right)^{\frac{n-2}{2}} \left( \frac{n}{n-t^2} \right)^{\frac{1}{2}} \rightarrow \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = \phi(t).
\]

As in Section 3.2 the limiting density is a standard Gaussian, and we can use Lebesgue’s Dominated Convergence Theorem to state the following conclusion. As \( n \to \infty \)

\[
\pi U^n \to^d N(0,1).
\]

We have agreement with the single-coordinate case of Theorem 1.1.
4 Uniform random variables on n-cubes

4.1 Geometry of Theorem 1.3

We can interprete Theorem 1.3 visually. It states that if we slice a cube, orthogonally to the diagonal, that in the limit of high-dimensions the area of the slices will be distributed normally along the diagonal. The following figures illustrate this interpretation in dimensions two and three. We see that the convergence is rapid.

(a) slicing orthogonally to the diagonal

(b) standardized length of the orthogonal line as a function of distance along the diagonal (graphed in blue), as compared to the standard Gaussian function (red)

Figure 2: slicing the square

4.2 Proof of Theorem 1.3

As part of our proof of Theorem 1.3 we will use the Central Limit Theorem. It is the Local Central Limit Theorem that will complete our eventual justification for the convergence on display in Figures 2 and 3. We state both Theorems here.

Theorem 4.1. (Central Limit Theorem) Let $X_1, X_2, ...$ be an infinite sequence of independent, identically distributed random variables, each with a mean $\mu$ and a finite, positive variance $\sigma^2$. The central limit theorem assures that as $n \to \infty$

$$S_n = \frac{\sum_{i=1}^{n} (X_i - \mu)}{\sqrt{n}\sigma} \to^d N(0,1)$$

Theorem 4.2. (Local Central Limit Theorem) When $S_n$ has a density denote it with $f_{S_n}(t)$. If $f_{S_n}(t)$ is bounded for some $N$ then as $n \to \infty$

$$f_{S_n}(t) \to \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$
Theorems 4.1 and 4.2 can be found in Petrov [2].

In order to prove Theorem 1.3 we need to show that if $U^n$ is a sequence of uniform random variables on centered unit $n$-cubes

$$C^n = \prod_{i=1}^{n} \left[ -\frac{1}{2}, \frac{1}{2} \right],$$

and with $\pi$ projecting onto the diagonal $(1, 1, \ldots, 1_n)$, and $T^n_c$ denoting

$$\|\pi U^n\|_2$$

that then

$$T^n_c \to^d N(0, 1/12).$$

Note that for a (measurable) subset $E \subseteq C^n$

$$P(U_n \in E) = \frac{L^n(E)}{L^n(C^n)} = L^n(E).$$

We would like to think of $U_n$ as a multivariate random variable,

$$U_n = (U_1, U_2, \ldots, U_n).$$

We then have

$$P(U_n \in E) = P((U_1, U_2, \ldots, U_n) \in E),$$
and $U_1$ through $U_n$ are uniform random variables. To see this let

$$E = \prod_{i=1}^{n} E_i,$$

where each $E_i$ is an interval in $[-1/2, 1/2]$. In other words, $E$ is a box in $C^n$. We have the following since multidimensional Lebesgue measure is the same as products of single dimensional Lebesgue measure (see Theorem 2.3).

$$P(U_1 \in E_1)P(U_2 \in E_2)\ldots P(U_n \in E_n) = P((U_1, U_2, \ldots, U_n) \in E) = P(U_n \in E) = L^n(E) = L^1(E_1)L^1(E_2)\ldots L^1(E_n).$$

By varying the length of $E_i$ we see that

$$P(U_i \in E_i) = L^1(E_i)$$

as long as $E_i$ is an interval. Then by the uniqueness criterion in Caratheodory’s Extension Theorem (2.4) we see that

$$P(U_i \in E_i) = L^1(E_i)$$

even if $E_i$ is a (measurable) set that is not an interval. In other words,

$$\forall i, U_i =_{d} U[-1/2, 1/2].$$

Furthermore, if we focus on the first equality (4.1) and use the same argument we see that we have independence. In conclusion, $U_1, U_2, \ldots, U_n$ are i.i.d. uniform random variables.

What happens when we project $U_n = (U_1, \ldots, U_n)$ onto the diagonal?

**Proposition 4.3.** The projection of $U_n = (U_1, \ldots, U_n)$ onto the diagonal $\{t^{(1,1,\ldots,1)}_{\sqrt{n}} : t \in \mathbb{R}\}$ will be denoted by $\pi U_n$. We have the following formula:

$$\pi U_n = \sum_{i=1}^{n} \frac{U_i}{\sqrt{n}} (1,1,\ldots,1) \frac{1}{\sqrt{n}}$$

**Proof.** Let $T_n$ denote $\sum_{i=1}^{n} \frac{U_i}{\sqrt{n}}$. The projection $\pi U_n$ must satisfy

$$\langle U_n - \pi U_n, \pi U_n \rangle = 0$$

$$\langle U_n, \pi U_n \rangle = \langle \pi U_n, \pi U_n \rangle$$

$$\langle U_n, \pi U_n \rangle = T_n^2$$

$$\frac{T_n}{\sqrt{n}} \sum_{i=1}^{n} U_i = T_n^2$$

$$\sum_{i=1}^{n} \frac{U_i}{\sqrt{n}} = T_n.$$
Proposition 4.3 gives a formula for $T_c^n$:

$$T_c^n = \frac{\sum_{i=1}^n U_i}{\sqrt{n}}.$$  

Note that the $T_c$ is nearly in the form of a standardized sum. For every $i$, $E(U_i) = 0$. It remains to compute the standard deviation:

$$E(U_i^2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = 2 \int_{0}^{\frac{1}{2}} x^2 dx = \frac{1}{12}.$$  

Thus,

$$\sqrt{E(U_i^2)} = \frac{1}{2\sqrt{3}}.$$  

The Central Limit Theorem then shows that

$$\frac{\sum_{i=1}^n U_i}{\frac{1}{\sqrt{3}} \sqrt{n}} \to_d N(0, 1).$$  

From this we conclude that

$$T_c^n \to_d N(0, 1/12).$$  

This completes the proof of Theorem 1.3.

5 Uniform random variables on the boundaries of n-cubes

Consider boundaries of centered, unit cubes:

$$F^n = \{ x \in C^n : (\exists i \in [1, n] : x_i = \pm 1/2) \text{ and } (\forall j \in [1, n] : j \neq i), -1/2 < x_j < 1/2 \}. $$  

There are $2n$ faces and thus

$$\text{area}(F^n) = 2n.$$  

**Definition 5.1.** A random variable $U$ is uniformly distributed on $F^n$ if for all (measurable) $E \subseteq R^n$ we have

$$P(U \in E) = \frac{\text{area}(E)}{(2n)}. $$  

Let $U^n$ be a sequence of random variables, each uniformly distributed on $F^n$, and let $\pi U^n$ be the sequence created by projecting each $U^n$ onto its cube’s diagonal $(1, 1, ..., 1_n)$. Define

$$T_b^n = \|\pi U_n\|_2.$$

We suspect that as $n \to \infty$, then $T_b^n$ will converge in distribution to a centered, normal random variable. We can gather evidence for this conjecture by simulating $T_b^n$ for large $n$.  


5.1 Simulations

We can simulate a random observation of $T^*_n$ as follows.

1. Choose one of the $2n$ faces of $F^n$.

2. Select a point on the face at random by generating $n - 1$ independent observations of $U(-1/2, 1/2)$.

3. Project the point onto the diagonal and compute its 2-norm.

In fact, due to symmetry of the eventual projection, we need only choose between two of the faces. This can be done by specifying $-1/2$ or $1/2$ for the first coordinate. Then the remaining coordinates can be determined randomly as described above, before projecting.

After repeating this procedure multiple times we then have a simulated sample of $T^*_n$.

This can be coded for with R. The following will generate a random sample of 100 observations of $T^{10}_b$.

```r
> A <- matrix(c(rbinom(100,1,.5)-.5,runif(900,-.5,.5)),nrow=100)
> t <- apply(A,1,sum)/sqrt(10)
```

![Histogram of t](image)

Figure 4: A histogram for a simulated sample of 100 observations of $T^{10}_b$
Figure 5: A Q-Q plot for a simulated sample of 100 observations of $T_{10}^b$

The Shapiro-Wilk normality test was also run to test the normality of the data. In this case the p-value was 0.3995. Thus, it seems we can’t yet rule out normality, even for $T_{10}^b$. It is certainly still possible that $T_{n}^b$ converges in distribution to a normal random variable. However, Figures 4 and 5 suggest that we shouldn’t expect a rate of convergence as fast as we saw with the cube (see Figure 3). After increasing $n$ from 10 to 1000 we observe the following.

Figures 6 and 7 give strong evidence that indeed $T_{n}^b$ converges in distribution to a normal random variable. In order to prove this fact, which is the conclusion of Theorem 1.4, we use the Lyapunov Central Limit Theorem.

5.2 Proof of Theorem 1.4

We start by stating Lyapunov’s Central Limit Theorem.
Figure 6: A histogram for a simulated sample of 1000 observations of $T_{b}^{1000}$

**Theorem 5.2.** (Lyapunov Central Limit Theorem) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables, each with a mean of zero and a finite third moment. Define $s_n^2 := \sum_{i=1}^{n} E(X_i^2)$. If Lyapunov’s condition

$$\lim_{n \to \infty} \frac{1}{s_n^3} \sum_{i=1}^{n} E(|X_i|^3) = 0$$

is satisfied then

$$\frac{\sum_{i=1}^{n} X_i}{s_n} \xrightarrow{d} N(0, 1).$$

The Theorem is applicable in our case. Looking back on our simulation procedure and using Proposition 4.3 we see that

$$T_b^n = B + \frac{\sum_{i=2}^{n} U_i}{\sqrt{n}},$$

where the $B$ stands for the binomial random variable

$$B \sim B\left(-\frac{1}{2}, \frac{1}{2}, .5\right),$$
Figure 7: A Q-Q plot for a simulated sample of 1000 observations of $T_{b}^{1000}$.

and $\forall i : i \in [2, n], \quad U_i \sim U(-1/2, 1/2)$.

We thus have

$$
E(|B|^2) = \frac{1}{4} \\
E(|U_i|^2) = \frac{1}{12} \\
E(|B|^3) = \frac{1}{8} \\
E(|U_i|^3) = \frac{1}{32},
$$

resulting in

$$
\frac{E(|B|^3) + \sum_{i=2}^{n} E(|U_i|^3)}{(E(|B|^2) + \sum_{i=2}^{n} E(|U_i|^2))^{\frac{3}{2}}} < (E(|B|^2) + \sum_{i=2}^{n} E(|U_i|^2))^{-\frac{1}{2}} \to 0,
$$

so Lyapunov’s condition is satisfied.
Also, as \( n \to \infty \) we have \( \frac{z_n}{\sqrt{n}} \to \frac{1}{2\sqrt{3}} \) determining the limiting variance. In summary,
\[
T_b^n = \frac{B + \sum_{i=2}^{n} U_i}{\sqrt{n}} \to^d N(0, \frac{1}{12}).
\]
This proves Theorem 1.4.

6 On the geometry of high-dimensional cubes

Given a sequence of centered, unit \( n \)-cubes
\[
C^n = \prod_{i=1}^{n} [-\frac{1}{2}, \frac{1}{2}],
\]
With each diagonal defined as
\[
\{x : x \in C^n \text{ and } \exists t : x = t(1,1,...,1)/\sqrt{n}\},
\]
We define orthogonal slices for \( t \neq 0 \) to be
\[
S^n_t = \{x : x \in C^n \text{ and } \langle x, t(1,1,...,1)/\sqrt{n}\rangle = 0\}.
\]
This allows us to define the following sequence of functions for \( t \neq 0 \):
\[
A_n(t) = \text{area}(S^n_t).
\]
\( \forall n, A_n(0) \) will be defined so that \( A_n \) is continuous at \( t = 0 \). \( A_n(t) \) measures areas of orthogonal slices of unit \( n \)-cubes. Is it true that \( \forall n \)
\[
\arg\max_{t \in \mathbb{R}} A_n(t) = 0?
\]

Our geometric intuition regarding cubes certainly suggests so much. The statement is in fact true, and we are in a good position to give a convoluted proof. We will use induction on the number of dimensions, \( n \). The hypothesis is true for \( n = 2 \) and \( n = 3 \) as can be seen directly (Figures 2 and 3). In both cases the max is obtained at zero. The induction step is then completed by remembering that the density for the sum of two independent random variables is the convolution of their two densities. In our case we have \( A_n(t) \) as the density function for
\[
T^n_c = \frac{\sum_{i=1}^{n} U_i}{\sqrt{n}},
\]
and \( A_{n+1}(t) \) as the density for
\[
T_{n+1} = \frac{\sum_{i=1}^{n+1} U_i}{\sqrt{n+1}} = \left( \frac{\sum_{i=1}^{n} U_i}{\sqrt{n}} + \frac{U_1}{\sqrt{n+1}} \right) \frac{\sqrt{n}}{\sqrt{n+1}}.
\]
So to go from \( A_n \) to \( A_{n+1} \) we first convolute with a scaled version of \( A_1 \) and then scale the result. Scaling doesn’t concern us so the following proposition completes the induction step.
Proposition 6.1. Let \( f_n(t) : \mathbb{R} \to \mathbb{R} \) be a function that is symmetric about zero, monotonically increasing left of zero, and monotonically decreasing right of zero. Then the convolution of \( f_n \) with \( A_1 \), namely \( f_{n+1}(s) = \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} f_n(t)dt \), has these same properties.

Proof. Let \( s_1 < s_2 \leq 0 \). Define the following domains of integration:

\[
S_2 = s_2 - \left[ \frac{1}{2}, s_2 + \frac{1}{2} \right] \\
S_1 = s_1 - \left[ \frac{1}{2}, s_1 + \frac{1}{2} \right] \\
I = S_1 \cap S_2 \\
U = S_2 - I \\
L = S_1 - I.
\]

We then have the following:

\[
f_{n+1}(s_2) - f_{n+1}(s_1) = \int_{S_2} f_n(t)dt - \int_{S_1} f_n(t)dt \\
= \int_{U} f_n(t)dt + \int_{I} f_n(t)dt - \int_{I} f_n(t)dt - \int_{L} f_n(t)dt \\
= \int_{U} f_n(t)dt - \int_{L} f_n(t)dt \\
> \int_{U} f(s_2 + \frac{1}{2}) - \int_{L} f(s_2 - \frac{1}{2}) \\
\geq 0.
\]

This shows that strict monotonicity is preserved left of zero. By the symmetry of \( f_n \), strict monotonicity is preserved right of zero as well. Finally, symmetry itself is preserved because

\[
f_{n+1}(s) = \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} f_n(t)dt = - \int_{-(s+\frac{1}{2})}^{-(s-\frac{1}{2})} f_n(-t)dt = \int_{-s-\frac{1}{2}}^{-s+\frac{1}{2}} f_n(t)dt = f_{n+1}(-s).
\]

\( \square \)
7 Hausdorff measure and the co-area formula

This section presents a more comprehensive approach to the problem of defining $k$-dimensional volume of sets in $\mathbb{R}^n$. We have been using Lebesgue measure and parametrizations to define top-dimensional $n$-volume and also $(n-1)$-volume, or area. However, we have been silent regarding $k$-volume for $0 < k < n - 1$. Furthermore, what if $k$ is a fraction? This calls for Hausdorff Measure.

For $0 < d \leq n$ we define the $d$-dimensional Hausdorff measure of subsets in $U \subseteq \mathbb{R}^n$.

First we define the diameter of a set via the euclidian distance function:

$$\text{diam}(U) = \sup\{\|y - x\|_2 : x, y \in U\}.$$ 

Now for a set $V \subseteq \mathbb{R}^n$ we define

$$H_d^\delta(V) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq V, \text{diam}(U_i) < \delta \right\}$$

This allows us to define the outer, d-dimensional Hausdorff measure.

**Definition 7.1.** The outer, $d$-dimensional, Hausdorff measure of a set $V \subseteq \mathbb{R}^n$ is defined to be

$$H_d(V) = \lim_{\delta \to 0} H_d^\delta(V).$$

It can be shown that this limit exists and that $H_d^\delta$ is an outer measure, and thus becomes a measure when restricted to the Caratheodory-measurable subsets of $\mathbb{R}^n$. We think of $H_d$ as measuring the $d$-volume of (measurable) sets.

Furthermore, we can normalize $H^n$ by including a constant

$$\frac{\text{volume}(B_1^d)}{2^d},$$

where sense is made out of volume($B_1^d$) through interpolation with the gamma function. Then in $\mathbb{R}^n$, $H^n$ will agree with $L^n$ and $H^{n-1}$ will agree with area as we have parametrically defined it, plus we will automatically have a definition for (say) the length of sets, or the fractional dimension of sets, etc.

Finally, as a substitute for Fubini’s Theorem we have the Co-area formula.

**Theorem 7.2.** Let $\Omega$ be an open set in $\mathbb{R}^n$ and $f$ a real-valued Lipschitz function on $\Omega$. Then for every $L^1$ function $g$,

$$\int_{\Omega} g(x)|\nabla f(x)|dH^n = \int_{-\infty}^{\infty} \left( \int_{f^{-1}(t)} g(x)dH^{n-1} \right) dt.$$

But, this is a bit heavy for dealing with our relatively simple sets. It could be useful though, if one wanted to expand on the theorems as presented in this paper.

For more information on Hausdorff Measure consult Federer [7].

26
References


