Ad: Math 650.
Para: Math 333 or 613
   Algebraic Number Theory.
Grading: 2 hw sets + final.
Contents:
1) Review gps, rings, fields, etc.

\[ 6 = 2 \cdot 3 = \left(1 + \sqrt{5}\right) \left(1 - \sqrt{5}\right) \]

2) Define alg. integers, examples.
   \[ \frac{a + b\sqrt{2}}{2} \]
   \[ 7 + 5i \]

3) Unique factorization - when it holds, \\
   when it fails.
   Unique factorization
   Gauss integers
   \[ \mathbb{Z}[i] \]
   \[ \mathbb{Z}[\sqrt{2}] \]
   \[ \mathbb{Z}[\sqrt{3}] \]

4) Units (in \( \mathbb{Z} \), here or units 2)
   \[ 1 \cdot \bar{1} = 1 \]
   \[ -1 \cdot \bar{1} = 1 \]
   \[ i \cdot \bar{i} = 1 \]
   \[ (-i) \cdot \bar{(-i)} = 1 \]
   \[ a + b\sqrt{2} \]
   \[ a + b\sqrt{3} \]
   \[ \text{find some units} \]
   \[ 3, 2\sqrt{2}, \sqrt{3}, 2\sqrt{3} \]

\[ 1 + 2 + 4 + 8 + 6 + \ldots \text{converge to 1} - 1 \]

is 2-adic number.

The sequence 3, 9, 27, 81, \ldots converge to 0
in 3-adic number.

3. Riemann zeta function.
   we'll show that the primes are 
   equidistributed among different residue 
   classes.

| \( \omega_1 \) | \( \omega_2 \) |
| 7 | 8 |
| 13 | 17 |
| 11 | 23 |
| 31 | 29 |
How big are $\mathbb{Z}$? odd integers? even integers? rationals $\mathbb{Q}$? compared to $\mathbb{N}$.

Ans: $\text{card}(\mathbb{Z}) = \text{card}(\text{even}) = \text{card}(\text{odd}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$
\[ \text{card}(\mathbb{R}) = \omega \neq \text{card}(\mathbb{N}) \]

\[ \text{card(algebraic)} = \text{card}(\mathbb{N}) \]

We believe \( \mathbb{N} \) cannot be put in bijection with \( \mathbb{R} \).

http://www.math.utah.edu/~klosin/550

So, let's assume \( \exists \) a bijection.

\[
\begin{align*}
1 & \rightarrow 0.1435143 \ldots \\
2 & \rightarrow 0.5971342 \ldots \\
3 & \rightarrow 0.7101115 \ldots \\
4 & \rightarrow 0.6000 \ldots \\
5 & \rightarrow 0.61616161 \ldots \\
6 & \rightarrow 0.015440478 \ldots \\
7 & \rightarrow 0.434794956 \ldots \\
\cdots \\
\mathbb{N} & \rightarrow \mathbb{R}
\end{align*}
\]

0.1121121 \ldots is not on the list \( b/c \) it doesn't appear in the list at 1st, 2nd, or 3rd digit.

Contradiction, so \( \text{card}(\mathbb{Q}) \neq \text{card}(\mathbb{N}) \).

So, also \( \mathbb{R} \) are "more numerous" than \( \mathbb{N} \).
$\mathbb{N} \not\to \mathbb{R}$

$\mathbb{N} \not\to \mathbb{Q}$

algebraic numbers $\to \exists$ transcendental
Countability of the alg. numbers

It is enough to show that the set of all polynomials with rational coefficients is countable.

Indeed, suppose we proved the set of polynomials is countable.

\[ \begin{align*}
1 & \rightarrow f_1 \\
2 & \rightarrow f_2 \\
3 & \rightarrow f_3 \\
4 & \rightarrow f_4 \\
5 & \rightarrow f_5 \\
6 & \rightarrow f_6 \\
\vdots & \end{align*} \]

The set of polynomials \( \text{w/rat. coeffs of degree } n \) is bijective to the set of \( n \)-element sequences with entries being rational.

\[ \begin{align*}
3x^2 + \frac{7}{2}x - 1 & \rightarrow (3, \frac{7}{2}, 1) \\
\frac{2}{3}x^2 + 5 & \leftarrow (\frac{2}{3}, 0, 5) \\
\end{align*} \]
Let's show there are countably many polynomials of deg 1 w/rational coefficients.

List rationals as: $d_1, d_2, d_3, d_4, \ldots$

$a_1x + d_1 \ a_1x + d_2 \ a_1x + d_3 \ \ldots$

$a_2x + d_1 \ a_2x + d_2 \ a_2x + d_3 \ \ldots$

$a_3x + d_1 \ a_3x + d_2 \ a_3x + d_3 \ \ldots$

\[ \vdots \]

So, the polyn. of deg $\leq 1$ are countable.