Q: How do we compare "sizes" of (infinite) sets?

If each element of the codomain is hit at most once then $f$ is an injection.

If each elt. of the codomain is hit at least once then $f$ is a surjection.

If each elt in co-domain is hit exactly once then $f$ is a bijection.

Note: For finite sets $A$ & $B$:
$A$ & $B$ have the same number of elements if and only if there is a bijection from $A$ to $B$.

Definition: Two sets $A$ & $B$ have the same size (cardinality) if there exists a bijection from $A$ to $B$. 
Q1: Are there more natural numbers than there are even natural numbers?

No.

\[ f(x) = 2x \]
\[ g(x) = \frac{1}{2} x \]

With this definition, you can have one set strictly contained in the other and they may still have the same cardinality.
More examples: \( \text{card}(\mathbb{N}) = \text{card}(\text{odd naturals}) \)

\[
\begin{align*}
\mathbb{N} &\xrightarrow{f} \text{odds} \\
f(x) &= 2x+1
\end{align*}
\]

\[
\begin{align*}
\text{card}(\text{even nat.}) &= \text{card}(\text{odd nat.}) \\
\text{even} &\xrightarrow{g} \text{odd} \\
g(x) &= x-1
\end{align*}
\]

Q2: Are there more integers than there are natural numbers (0 \( \notin \) \( \mathbb{N} \))? No.

\[
\begin{array}{cccccccc}
\mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
\mathbb{Z} & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \ldots
\end{array}
\]

\[
\begin{align*}
f(21) &= -10 \\
f(14) &= 7 \\
f(x) &= \begin{cases} 
\frac{x}{2} & \text{if } x \text{ even} \\
0 & \text{if } x = -1 \\
\frac{x+1}{2} & \text{if } x \text{ odd}
\end{cases}
\end{align*}
\]

\[
\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})
\]

Def: Let \( A \) be a set. If \( \text{card}(A) = \text{card}(\mathbb{N}) \), we call \( A \) countable.
Q3: Are there more positive rationals than there are natural numbers?

\[ \mathbb{N} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \ldots \]

\[ \mathbb{Q} \quad \frac{1}{1} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{2}{3} \quad \frac{3}{1} \quad \frac{3}{2} \quad \frac{3}{4} \quad \frac{4}{1} \quad \frac{4}{2} \quad \frac{4}{3} \quad \frac{4}{5} \quad \frac{5}{1} \quad \frac{5}{2} \quad \frac{5}{3} \quad \frac{5}{4} \quad \frac{5}{6} \quad \frac{6}{1} \quad \frac{6}{2} \quad \frac{6}{3} \quad \frac{6}{4} \quad \frac{6}{5} \quad \frac{6}{7} \quad \ldots \]

\[ \mathbb{Q}^+ \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{3}{2} \quad \frac{2}{3} \quad \frac{2}{4} \quad \frac{3}{4} \quad \frac{4}{3} \quad \frac{4}{2} \quad \frac{4}{1} \quad \frac{5}{2} \quad \frac{5}{3} \quad \frac{5}{4} \quad \frac{5}{6} \quad \frac{6}{5} \quad \frac{6}{4} \quad \frac{6}{3} \quad \frac{6}{2} \quad \frac{6}{1} \]

No. \quad \text{card}(\mathbb{Q}^+) = \text{card}(\mathbb{N})
What about the reals?

The decimal expansion of a rational is periodic from some point on.

\[ 1.78512121212 \ldots \in \mathbb{Q} \]

\[ = 1.785 + 0.000121212 \ldots \]

\[ \begin{array}{c}
\frac{1785}{1000} \\
0.00012 + 0.0000012 + 0.000000012 + \cdots
\end{array} \]

\[ \frac{1 - r}{1 - r} = \frac{0.00012}{1 - 10^{-2}} = \frac{12}{10^5 - 10^3} \]

\[ \alpha \in \mathbb{Q} \iff \alpha \text{ has a periodic decimal expansion from some point on.} \]

Idea: Let's assume there is a bijection \( f: \mathbb{N} \rightarrow \mathbb{R} \) and get a contradiction.