Algebraic & transcendental #s.

Integers: $0, \pm 1, \pm 2, \ldots \in \mathbb{Z}$

Rationals: $\frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$ ($\mathbb{Q}$)

Real numbers: $\sqrt{\frac{a}{b}}$ if $pb = aq$

$3 \leq \{3, 3.1, 3.14, 3.141, \ldots \} \leq \pi$

$\pi \approx 3.142$ or $3.1419$

Complex number: "real + s/v.to $x^2 + 1 = 0$".

Irrationals are: a real number which is not rational.

$\sqrt{2}, e, \pi, \sqrt{3}, \sqrt{7}$

Easy proof
Def: A real number is called 
ALGEBRAIC if it is a root
of a polynomial with rational
coefficients.

\[ x - 1 = 0 \]
\[ 1 \text{ satisfies this poly.} \]
\[ \text{so } 1 \text{ is an alg. number} \]

Alg. numbers
\[ x^3 - 2 \quad 2^{\frac{1}{3}} \text{ alg.} \]
\[ x \quad 0 \text{ alg.} \]
\[ 4x - 1 \quad \frac{1}{4} \text{ alg.} \]
\[ x - \alpha \quad \alpha \in \mathbb{Q} \text{ alg.} \]
\[ x^2 - 2 \quad \sqrt{2} \text{ alg.} \]
\[ x^2 - 7 \quad \sqrt{7} \text{ alg.} \]
\[ x^2 - 3 \quad \sqrt{3} \text{ alg.} \]
\[ e \text{ is not alg. (Hermite 1873)} \]

Def: A real # that is not algebraic
is called transcendental.

\[ \pi \text{ is transcend. (Lindemann 1882)} \]

\[ \sum_{n=1}^{\infty} 10^{-n!} = 0.1100010101010\ldots \]

\[ \frac{\pi}{2} \]

6/ a rational number can
be recognized from its decimal expansion
as this expansion must be periodic
from some point on.

Project: Prove that \( \sum_{n=1}^{\infty} 10^{-n!} \) is transc.
Which of these are known to be transcendental?

\[ \sum 10^{-n!} \checkmark \]

\[ e, \quad (1873) \checkmark \]

\[ \pi, \quad (1882) \checkmark \]

Hilbert: Is it true that if \( a \) is an algebraic \((a \neq 0,1)\) and \( b \) is irrational and algebraic, then \( a^b \) is transcendental.

Answer: Yes (Gelfond-Schneider 1934)

\[ 2^{\sqrt{2}} \text{ is transc.} \quad \sqrt{2}^{\sqrt{2}} \text{ is transc.} \]

\[ 3^{\sqrt{2}} \text{ is transc.} \]

\[ 2^\pi, \quad 7^\pi \text{ is transc.} \]

Examples:

\[ e^a \quad (a = \text{alg, } a \neq 0) \text{ is transc.} \]

\[ e^\pi \quad (\text{Gelfond-Schneider}) \text{ is transc.} \]

\[ \sin a, \cos a, \tan a \text{ are transc.} \]

\[ \text{Not known: } \pi + e, \pi - e, \pi e, \pi^e, \pi, \pi^e, \pi^e, \sum (3) \]

\[ \Gamma \left( \frac{1}{4} \right) \]