\[
N \subset Z \subset Q \subset \text{real alg.} \subset R
\]

have same cardinality (countable)

WANT THIS

these are "bigger" (not countable)

If we can show that the real algo. numbers are countable, then we know that not all reals are algebraic. Hence \(\exists\) transcendental numbers.
Last time we said that ETS (enough to show)

Jorge Luis Borges "Library Babel"

there are only countably many polynomials with rational coefficients.

To show $\exists$ countably many polys w/ coeffs $\in \mathbb{Q}$ we proceed in steps.

1. $\exists$ only countably many polys w/ rational coeffs of degree $\leq 1$.

Pf: We know $\exists$ only countably many rational.

So, let's number them: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots$

\[
\begin{array}{cccc}
\alpha_1x + d_1 & \alpha_2x + d_2 & \alpha_3x + d_3 & \ldots \\
1 & 2 & 3 & 4 \\
\end{array}
\]
2. There are only countably many polynomials with rational coefficients of degree \( \leq 2 \).

**Proof:** We know there are countably many rationals: \( d_1, d_2, d_3, \ldots \) and by (1) we know there are only countably many polynomials of degree 1 with rational coefficients. Let's number them:

\[ \beta_1, p_2, p_3, \ldots \]

\[ \alpha_1 x^2 + \beta_1, \alpha_2 x^2 + \beta_2, \alpha_3 x^2 + \beta_3, \ldots \]

\[ \alpha_1 x^2 + \beta_2, \alpha_2 x^2 + \beta_2, \alpha_3 x^2 + \beta_2, \ldots \]

\[ \alpha_1 x^2 + \beta_3, \alpha_2 x^2 + \beta_3, \alpha_3 x^2 + \beta_3, \ldots \]

Let \( n \in \mathbb{Z}^+ \).

(3) There are only countably many polynomials of degree \( n \) with rational coefficients.

**Proof:** Induction. \( \square \)
(4) There are only countably many polynomials with rational coefficients.

Proof:
\[
\begin{align*}
\alpha_1 & \quad \alpha_2 & \quad \alpha_3 & \quad \alpha_4 & \quad \ldots & \quad \text{(rationals)} \\
\beta_1 & \quad \beta_2 & \quad \beta_3 & \quad \beta_4 & \quad \ldots & \quad \text{(deg 1 poly)} \\
\delta_1 & \quad \delta_2 & \quad \delta_3 & \quad \delta_4 & \quad \ldots & \quad \text{(deg 2 poly)} \\
\gamma_1 & \quad \gamma_2 & \quad \gamma_3 & \quad \gamma_4 & \quad \ldots & \quad \text{(deg 3 poly)}
\end{align*}
\]
If $A$ is a finite set, add 1 element to it & you get a bigger set.

But suppose $A$ is countable (i.e., in bijection with $\mathbb{N}$). How hard is it to enlarge $A$?

- add 1 element:

\[
\begin{align*}
\vdots \quad a_3, a_4, \ldots \quad &\rightarrow \quad A \\
X_1, X_2, \ldots \quad &\rightarrow \quad X
\end{align*}
\]

- no change in card.

- add any finite amount of elements:

\[
\begin{align*}
a_1, a_2, a_3, a_4, a_5, \ldots \quad &\rightarrow \quad A \\
x_1, x_2, \ldots \quad &\rightarrow \quad X
\end{align*}
\]

\[
\text{card} (A \cup X) = \text{card}(A)
\]

- add a countable number of elements in $A$:

\[
\begin{align*}
\vdots \quad a_3, a_4, \ldots \quad &\rightarrow \quad A \\
x_1, x_2, \ldots \quad &\rightarrow \quad X
\end{align*}
\]

\[
\text{card} (A \cup X) = \text{card}(A)
\]

- Add countably many countable sets (take the union of countably many countable sets):

\[
\begin{align*}
a_1, a_2, a_3, \ldots \quad &\rightarrow \quad A \\
x_1, x_2, \ldots \quad &\rightarrow \quad X_1
\end{align*}
\]

\[
\begin{align*}
x_2, x_3, \ldots \quad &\rightarrow \quad X_2
\end{align*}
\]

\[
\vdots
\]

\[
\text{card} (\bigcup A \cup X_1 \cup X_2 \cup \ldots) = \text{card}(A)
\]

One way to get a bigger set than $A$ is to take the set of all subsets of $A$. 

Theorem: e is irrational

Proof:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots$$

$$0 < e - S_n$$

$$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots$$

$$\frac{1}{(n+1)!} \left[ 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots \right]$$

$$\frac{1}{1 - \frac{1}{n+1}} = \frac{1}{\frac{n+1-1}{n+1}} = \frac{n+1}{n}$$

$$0 < e - S_n < \frac{1}{n!}$$

$$0 < n! [e - S_n] < \frac{1}{n}$$
\[ p+2: \]

Assume \( e \) is rational

\[ e = \frac{p}{q} \quad \text{where} \quad p, q \in \mathbb{N} \quad \text{and} \quad \gcd(p, q) = 1 \]

Pick \( n > q \) \( \Rightarrow \frac{1}{n} > \frac{1}{q} \) \( \Rightarrow \left( \frac{1}{n} < 1 \right) \)

\( 0 < n! \left[ e - \frac{1}{n} \right] < \frac{1}{n} < 1 \)

Not an integer!

\[ n! \left[ \frac{p}{q} - \left( \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \right) \right] \]

\[ n! \left[ \frac{p}{q} - \left( \underbrace{n! + n! \cdot \frac{n!}{2^{n-2}} \ldots + \frac{n!}{n^n}}_{\text{integer}} \right) \right] \Rightarrow \text{Integer} \]

\[ \frac{n!}{3} = \left( \frac{7.6.5.4.3.2.1}{3} \right) \]

Contradiction

\[ \therefore \quad e \text{ must be irrational} \]

\( \Box \)
\[ u_j b \]

\[ p(a \text{ is divisible by } p) = \frac{1}{p} \]  
\[ p(a \text{ and } b \text{ are BOTH divisible by } p) = \frac{1}{p^2} \]  
\[ p(a \text{ and } b \text{ are not both divisible by } p) = 1 - \frac{1}{p^2} \]

\[ f_{p_{\text{rel}}} = (1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})... \]

**ASIDE**

\[ \frac{1}{1-x} \text{ for } |x|<1 \]

\[ (1 - \frac{1}{p^2})^{-1} = (\frac{p^2-1}{p^2})^{-1} = \frac{1}{1-\frac{1}{p^2}} = \frac{1}{1-\frac{1}{p^2}} \]

\[ = \frac{1}{1-\frac{1}{p^2}} \]

\[ \text{for } p^2 \frac{1}{1-x} = 1 + \frac{1}{x} + \frac{1}{x^2} \ldots \]

\[ \rho = \left( (1 + \frac{1}{x} + \frac{1}{x^2})(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots) \right)^{-1} \]

\[ \rho = \left( 1 + \frac{1}{3x} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} \ldots \right)^{-1} \]

\[ \rho = \frac{\sqrt[n]{\frac{1}{y^2}}}{\frac{1}{y^2}} \rightarrow \left( 2 \right)^{-1} \]

\[ = \frac{6}{11^2} \approx 0.06 \]