Math 3220, Spring 2000

Facts about integration

Let $[a, b]$ be a closed interval. There is a class of function f on $[a, b]$ called *intergrable function* to which one can associate a unique real number

$$\int_a^b f(t)dt$$

called *the integral of f over* $[a, b]$ satisfying the following properties:

1. Constant functions are integrable and if $f(t) = k$, then $\int_a^b f(t)dt = k(b-a)$.
2. If k is a constant f and g are integrable on $[a, b]$ then so are kf and $f + g$ and
$$\int_a^b kf(t)dt = k\int_a^b f(t)dt$$
and
$$\int_a^b (f+g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt.$$
3. If f and g are integrable and $f(t) \leq g(t)$ for all t then
$$\int_a^b f(t)dt \leq \int_a^b g(t)dt.$$
4. If f is integrable on $[a, b]$ then so is $|f|$ and
$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$$
5. If $a \leq c \leq b$ and f is a function on $[a, b]$, then f is integrable on $[a, b]$ if and only if its is integrable over both $[a, c]$ and $[b, c]$, in which case
$$\int_a^b (f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Since we are mainly interested in continuous functions, the followign main theorem is extrememly useful:

Theorem 1 *If f is continupous on $[a, b]$ then it is integrable over $[a, b]$.*

Note however, that not all integrable functions are continuous.

We have the following truly remarkable idea:

Theorem 2 *Let f be integrable on $[a, b]$. Define a new function F on $[a, b]$ by the rule*

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$. Furthermore, if f is continuous at $x_0 \in (a, b)$, the F differentiable atx_0, and

$$F'(x_0) = f(x_0).$$