Math 3210-2, Fall 1999

Test III Solution

1. For all n, let f_n be the function with domain $\mathbf{R}$ given by

$$f_n(x) = \begin{cases} \frac{x}{n}\sin\left(\frac{n}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) Prove that each f_n is continuous on $\mathbf{R}$.

Proof: compositions and products of continuous functions are continuous on their domains. Now the constant functions $1/n$, the rational function $x \mapsto n/x$ and the sine function are all continuous on their domains, so each f_n is continuous on $\mathbf{R} - \{0\}$.

To check that each f_n is continuous at 0, observe that since the sine functions is bounded by 1, we have that

$$-|x|/n \leq f_n(x) \leq |x|/n$$

for all $x \in \mathbf{R}$. Thus, if (x_k) is a sequence of real numbers converging to 0, we have

$$-|x_k|/n \leq f_n(x_k) \leq |x_k|/n.$$

But since (x_k) converges to 0, so does $(|x_k|)$, and therefore, by the squeeze theorem, $(f_n(x_k))$ converges to $0 = f_n(0)$. Thus, f_n is continuous at 0 too.

(b) Let $M \geq 0$ and let $f(x) = 0$ for all $x \in \mathbf{R}$. Prove that (f_n) converges to f uniformly on $[-M, M]$.

Proof: In the previous part, we saw that for all $x \in \mathbf{R}$ and for all n,

$$-|x|/n \leq f_n(x) \leq |x|/n.$$

On $[-M, M]$, we have that $|x| \leq M$, so the above shows that

$$-M/n \leq f_n(x) \leq M/n$$

for all $x \in [-M, M]$, and all n.

To prove uniform convergence, let $\varepsilon > 0$. Choose $N \geq M/\varepsilon$. Then if $n > N$ and $x \in [-M, M]$,

$$|f_n(x) - 0| \leq M/n < \varepsilon.$$

So (f_n) converges to f uniformly on $[-M, M]$.

(c) Does (f_n) converge to f uniformly on $\mathbf{R}$? Why or why not?

SOLUTION: The convergence *is not* uniform on $\mathbf{R}$. To see this, observe that

$$f_n(2n/\pi) = 2/\pi$$

for all n. Thus for $\varepsilon = 2/\pi$, for all n there exists $x \in \mathbf{R}$ such that

$$|f_n(x) - 0| \not< \varepsilon,$$

so the convergence can't be uniform.

2. **Do one out of three.**

(a) Let the function f be continuous on an interval I and suppose that $f(x)$ is a rational number for all $x \in I$. Prove that f is a constant function.

PROOF: Suppose not. Then there exist $a < b \in I$ such that $f(a) \neq f(b)$. Choose any irrational number y between $f(a)$ and $f(b)$. By the intermediate value theorem, there exists $c \in (a.b)$ such that $f(c) = y$, contradicting the hypothesis that f takes only rational values.

(b) Let f be a function defined on $\mathbf{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$, and suppose that f is continuous at 0. Prove that f is continuous on $\mathbf{R}$.

PROOF: First observe that the condition $f(x+y) = f(x) + f(y)$ implies $f(0) = 0$ (set $x = y = 0$) and then that $f(-x) = -f(x)$ (set $y = -x$).

Let $x_0 \in \mathbf{R}$ and let (x_n) be a sequence of real numbers converging to x_0. Then $(x_0 - x_n)$ is a sequence converging to 0, so by continuity at 0, the sequence of values

$$(f(x_0 - x_n))$$

converges to $f(0) = 0$.

But using the hypothesis that f preserves sums, we have that

$$(f(x_0 - x_n)) = (f(x_0) - f(x_n))$$

so $f(x_n)$ converges to $f(x_0)$. Thus, f is continuous at x_0.

(c) Let f be defined on some set S and suppose that f is continuous at the point $x_0 \in S$. Suppose $f(x_0) > 0$. Prove that there exists $\delta > 0$ such that $f(x) > 0$ whenever $x \in S$ and $|x - x_0| < \delta$.

PROOF: Set $\varepsilon = f(x_0)$. By the ε-δ definition of continuity, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies that

$$|f(x) - f(x_0)| < \varepsilon = f(x_0),$$

i.e., that

$$-f(x_0) < f(x) - f(x_0) < f(x_0),$$

so that

$$0 < f(x) < 2f(x_0).$$

3. Indicate whether each statement is true or false.

   (a) The series $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$ converges.
   True: On the face of it, this is not an alternating series. But if we let $s_n = \frac{1}{2n-1} + \frac{1}{2n}$, then

   $$\sum_{n=1}^{\infty} (-1)^{n+1} s_n = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) - \left(\frac{1}{7} + \frac{1}{8}\right) + \cdot$$

   Since $s_{n+1} s_n$ for all n, this series converges by the alternating series test.

   (b) If functions f and g defined on $(-1, 1)$ are discontinuous at x_0, then the product function fg is discontinuous at x_0.

   False: Let f be the function give by $f(x) = 0$ if $x \neq 0$ and $f(0) = 1$ and let g be the function given by $g(x) = 1$ is $x \neq 0$ and $g(0) = 0$. Then neither f nor g is continuous at 0, yet the product fg is just the zero function, which is continuous everywhere.

   (c) If a sequence of functions (f_n) converges uniformly on S to the function f, then f is continuous on S.

   False: take any function f on S which is not continuous on S. Set $f_n = f$ for all n. The the constant sequence of functions (f_n) converges uniformly to f on S.

   (d) Let f be defined and continuous on $[0, +\infty)$ and suppose f is *not* bounded. Then there exists $x_0 \geq 0$ such that $|f(x_0)| \leq |f(x)|$ for all $x \geq 0$.

   False: An exact example is not entirely simple to produce. Something like

   $$f(x) = x + 1 + \frac{(x+1)^2}{x+2} \sin(x+1)$$

   should be a counterexample.

   (e) Let f be defined on $(-1, 1)$; if f is continuous at 0, then $\lim\limits_{x \to 0} f(x)$ exists.

   True: This follows immediately using either the sequential definition of limits and continuity or the ε-δ definition.