Math 3210-2, Fall 1999

Test II Solution

1. Unfortunately, in writing this solution, I discovered a small error in the statement of the problem. Below is the correct statement and a solution. I will make appropriate adjustments when grading the problem.

   Prove the following fact: If $A \subset \mathbf{R}$ is bounded above, and $s = \sup A$, then for all $\varepsilon > 0$, there exists $x \in A$ such that

   $$s - \varepsilon < x \leq s.$$

   Proof. We proceed by contradiction. Let $\varepsilon > 0$ and suppose there does not exist $x \in A$ such that $s - \varepsilon < x \leq s$. Since s is an upper bound for A, we have $x \leq s$ for all $s \in A$ and therefore, by our supposition,

   $$x \leq s - \varepsilon \text{ for all } x \in A.$$

   But this says that $s - \varepsilon$ is an upper bound for A, and since s is the least upper bound, it must be that

   $$s \leq s - \varepsilon,$$

   which contradicts that $\varepsilon > 0$.

2. Fix a number a with $0 \leq a \leq 1$. Consider the sequence (s_n) defined recursively by

   $$s_1 = a \text{ and } s_{n+1} = \sqrt{(s_n + 1)/2}$$

   (a) Show that $0 \leq s_n \leq 1$ for all n.

   Proof. Well, $0 \leq s_1 = a \leq 1$, so the case $n = 1$ works. Now suppose that $0 \leq s_k \leq 1$. Then

   $$\frac{1}{2} \leq \frac{s_k + 1}{2} \leq 1$$

   and hence

   $$\sqrt{\frac{1}{2}} \leq \sqrt{\frac{s_k + 1}{2}} \leq 1.$$

   Thus, $0 \leq s_{k+1} \leq 1$. So by the Principle of Mathematical Induction, $0 \leq s_n \leq 1$ for all n.

(b) Prove that (s_n) converges.

PROOF. In the previous part we have shown that (s_n) is a bounded sequence, so it suffices to prove that it is monotone. To see this, consider the expression $s_{n+1} - s_n$. Since all term of our sequence are non-negative,

$$s_{n+1} - s_n \geq 0 \text{ if and only if } s_{n+1}^2 - s_n^2 \geq 0.$$

But

$$s_{n+1}^2 - s_n^2 = (s_n + 1)/2 - s_n^2 = \frac{1}{2}(1 + 2s_n)(1 - s_n).$$

Thus, this expression is non-negative whenever $-1/2 \leq s_n \leq 1$, which is always the case by the previous part. So we have a non-decreasing sequence, which is bounded above by 1, so it converges.

3. Recall that a series *converges absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges.

Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Show that

$$\sum_{j=1}^{\infty} a_{k_j}$$

converges absolutely for any subsequence (a_{k_j}) of (a_k).

PROOF. Let (s_n) be the sequence of partial sums of $\sum_{k=1}^{\infty} |a_k|$ and let (t_i) be the sequence of partial sums of $\sum_{j=1}^{\infty} |a_{k_j}|$. Then

$$\begin{aligned} t_i &= |a_{k_1}| + |a_{k_2}| + \cdots + |a_{k_i}| \qquad (1) \\ &\leq |a_1| + |a_2| + \cdots + |a_{k_i}| = s_{k_i}. \qquad (2) \end{aligned}$$

Now the sequence (s_{k_i})—being a subsequence of the convergent sequence (s_n)—converges, and hence is bounded above. The above inequality then shows that (t_i) is bounded above as well. Since it is the sequence of partial sums of a series with non-negative terms, it is a non-decreasing sequence, hence converges. This means precisely that $\sum_{j=1}^{\infty} a_{k_j}$ converges absolutely.

4. True or false.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$ converges.

TRUE. Using the ratio test: if $a_k = \frac{2^k k!}{k^k}$, then

$$\frac{a_{k+1}}{a_k} = 2\left(\frac{k}{k+1}\right)^k$$

which approaches $2/e < 1$ as $k \to \infty$.

(b) Every sequence has a lim sup and a lim inf.

TRUE. Keep in mind that ∞ and $-\infty$ are allowed.

(c) If (s_n) is a bounded sequence and (s_{n_k}) a subsequence, then $\lim_{k\to\infty} s_{n_k}$ exists.

FALSE. A bounded sequence will have some convergent subsequences. But for example, $((-1)^k)$ is a subsequence of the bounded sequence $((-1)^n)$, so not every subsequence converges.

(d) If $\sum a_k$ converges and $\sum b_k$ diverges, then $\sum(a_k + b_k)$ diverges.

TRUE. The sequence of partial sums for $\sum(a_k + b_k)$ is the sum of the sequences of partial sums for $\sum a_k$ and $\sum b_k$. Thus, the statement reduces to: If (s_n) converges and (t_n) diverges, then $(s_n + t_n)$ diverges. To prove this, suppose by contradiction that $(s_n + t_n)$ converges. Then since $t_n = (s_n + t_n) - s_n$, the limits laws would imply that (t_n) converges, contradicting a hypothesis.