Math 3210-2, Fall 1999

Test I Solution

1. Let (s_n) be a sequence.

   (a) Suppose (s_n) converges to s. Then $(|s_n|)$ converges to $|s|$.

   PROOF. Let $\varepsilon > 0$. Since (s_n) converges to s, there exists N such that $|s_n - s| < \varepsilon$ whenever $n \geq N$. Thus, if $n \geq N$,

   $$||s_n| - |s|| \leq |s_n - s| < \varepsilon,$$

   where the first inequality was established in HW1. It follows that $(|s_n|)$ converges to $|s|$.

   (b) Suppose that $(|s_n|)$ converges to 0. Then (s_n) converges to 0 as well.

   PROOF. We have

   $$-|s_n| \leq s_n \leq |s_n|$$

   for all n. Since both outside sequences converge to 0, so does the inside one—(s_n)—by the squeeze theorem.

   (c) Suppose that $(|s_n|)$ converges to $s \neq 0$. Give an example to show that (s_n) needn't converge.

   EXAMPLE. Let $s_n = (-1)^n$. We have seen that (s_n) diverges. On the other hand, $|s_n| = 1$ for all n so that $(|s_n|)$ converges to 1.

2. be a convergent sequence and let $\varepsilon > 0$. Then there exists N such that if $m, n > N$,

   $$|s_n - s_m| < \varepsilon.$$

   PROOF. Let $\varepsilon > 0$. Since (s_n) converges—call the limit s—there exists an N such that

   $$|s_n - s| < \frac{\varepsilon}{2}$$

   whenever $n > N$. Therefore, if $n, m > N$,

   $$|s_n - s_m| = |s_n - s - (s_m - s)| \leq |s_n - s| + |s_m - s)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

3. Let (s_n) be a sequence such that $\lim s_n = -\infty$ and let (t_n) be a sequence which is bounded above (meaning there exists M such that $t_n < M$ for all n). Then

$$\lim(s_n + t_n) = -\infty.$$

PROOF. Let $L \in \mathbf{R}$. Since (s_n) diverges to $-\infty$, there exists and N such that for all $n \geq N$,

$$s_n < L - M.$$

Therefore, since since $t_n < M$, we have

$$s_n + t_n < L - M + M = L.$$

We have proved that $\lim(s_n + t_n) = -\infty$.

4. Indicate whether the given statement is true or false.

   (a) If a, b, $c \in \mathbf{R}$ and $ac = bc$, then $a = b$.
   FALSE. If $c = 0$, then for any a, b, we have $ac = bc$.

   (b) If $S = \{x : x^2 < 2\}$, then S is bounded below.
   TRUE. $x^2 < 2$ is equivalent to $-\sqrt{2} < x < \sqrt{2}$.

   (c) $\lim(\sqrt{n^2 + 3n - 1} - n) = 3/2$.
   TRUE. Rationalizing the numerator, we find that

$$\sqrt{n^2 + 3n - 1} - n = \frac{3n - 1}{\sqrt{n^2 + 3n - 1} + n}.$$

   Dividing numerator and denominator by n, we obtain

$$\frac{3 - 1/n}{\sqrt{1 + 3/n - 1/n^2} + 1}$$

   which converges to $3/2$ by the limit laws.

   (d) If (s_n) converges to s, the $\sup\{s_n\} = s$.
   FALSE. For example, the sequence $(1/n)$ converges to 0 but $\sup\{1, 1/2, 1/3, \ldots\} = 1$.

   (e) If (s_n) is a bounded sequence, then it converges.
   FALSE. The sequence $1, -1, 1, -1, 1, \ldots$ is bounded but diverges.