Solution to problem 3.8

Let me try to set out everything I can about problem 3.8.

The logical structure:

we are to prove a statement of the form

$$\text{For } a,\, b \in \mathbf{R}, \text{ iff } S \text{ then } T,$$

where S is the statement

$$A \leq b_1 \text{ for all (real numbers) } b_1 > b$$

and T is the statement

$$a \leq b.$$

In other words, we need to show that if a and b are real numbers such that S is true about them, then we need to show that T is true about them.

Let us examine S more closely. For a given a and b, it is a statement of the form for $b_1 \in \mathbf{R}$,

$$\text{if } S_1 \text{ then } S_2,$$

where S_1 is the statement

$$b_1 > b$$

and S_2 is the statement

$$a \leq b_1.$$

What is the opposite of S? It is the statement that there exists *one* $b_1 \in \mathbf{R}$ such that S_1 does not imply S_2, i.e., one b_1 for which S_1 holds but S_2 does not.

The proof:

suppose that a and b are real numbers such that S holds, i.e., such that $a \leq b_1$ whenever $b_1 > b$. This is our hypothesis. We want to show that T holds.

Assume it does not, i.e., assume that $a > b$. Then define

$$b_1 := \frac{a-b}{2}.$$

Given that $a > b$, we can show using the various A1–A4, M1–M4, DL and O1–O5, that

$$a > b_1 > b.$$

Therefore, we have found a b_1 for which S_1 hold, but S_2 does not hold, which is the opposite of S. In other words, by making the assumption $a > b$, we have arrived a the opposite of our hypothesis. This is a contradiction, so the assumption $a > b$ must have been invalid, so $a \leq b$.

The proof again:

I now give the proof without any reference to S, T, etc.

Let $a, b \in \mathbf{R}$ and suppose that $a \leq b_1$ for all reals $b_1 > b$. We will prove that $a \leq b$. Suppose not, i.e., suppose that $a > b$. Set

$$b_1 := \frac{a-b}{2}.$$

Then

$$a > b_1 > b,$$

which contradicts our hypotheses. Therefore,

$$a \leq b.$$