

## Math 2250-1 Maple Project 4 Summer 2009

**Directions:** Hand in a single Maple document, printed and stapled neatly, which contains the answers to the exercises below. At the top of this document, you should create a text field with an appropriate title, the date, your name, and UID. Below this header, please answer the exercises in order. If an exercise calls for computations by hand, you can type up your solutions in a text field, or leave enough space so that you can hand-write your computation/explanation after printing the document.

**Maple Help:** Maple has an extensive Help menu! You can search for commands to find syntax, related commands, and extensive examples. Additionally, there are some introduction to Maple guides posted on our class webpage ([www.math.utah.edu/~kitchen/2250Sum09](http://www.math.utah.edu/~kitchen/2250Sum09)). You can also get help from tutors in the tutoring center, and follow the guidelines in the book as necessary.

### Exercises, Part 1:

The section 7.1 Application starting on page 405 of the text describes Kepler's Laws of planetary motion. These laws imply that the position vector  $\mathbf{r}(t) = (x(t), y(t))$  for a planet satisfies a second order differential equation of the form

$$\mathbf{r}'' = -\frac{k\mathbf{r}}{r^3} \quad (1)$$

where  $k$  is a constant and  $r = \sqrt{x^2 + y^2}$  is the distance from the planet to the sun. This is the same as a non-linear second order system of differential equations:

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \frac{kx}{\sqrt{x^2+y^2}^3} \\ \frac{ky}{\sqrt{x^2+y^2}^3} \end{bmatrix} \quad (2)$$

However, one can convert this system to a single second order differential equation (see page 406 of the text)

$$\frac{d^2z}{d\theta^2} + z = \frac{k}{h^2} \quad (3)$$

where  $h$  is another constant and  $z = \frac{1}{r}$  as a function of  $\theta$  (recall we can convert cartesian coordinates  $(x, y)$  to polar coordinates  $(r, \theta)$  with  $r = \sqrt{x^2 + y^2}$  as above and  $\tan \theta = \frac{y}{x}$ ). By solving this equation, we can find  $r(\theta)$ —that is the position of a planet as a polar function. This answer is desirable because we can, for example, graph such a function.

### Problems:

1. By hand, separate the second order differential equation  $z'' + z = \frac{k}{h^2}$  into a nonhomogeneous  $2 \times 2$  system of first order differential equations. Then, express your system in matrix form – i.e. as

$$\frac{d}{d\theta} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \mathbf{f} \quad (4)$$

where  $\mathbf{A}$  is the  $2 \times 2$  coefficient matrix of your system.

2. Use **Maple** to find the eigenvalues and eigenvectors of **A**. Apply **Theorem 1** on page 419 to obtain the complementary solution to  $z'' + z = \frac{k}{h^2}$ . Use the method of undetermined coefficients (by hand) to find the particular solution, then write the general solution by hand, or type it into your **Maple** worksheet. Check your answer against equation (11) on page 407. You should have a solution of the form

$$z(\theta) = A \cos \theta + B \sin \theta + \frac{k}{h^2} \quad (5)$$

3. From this solution, we can deduce using the definition  $z = \frac{1}{r}$  and some trig identities that  $r(\theta)$  takes the form

$$r(\theta) = \frac{L}{1 + e \cos(\theta - \alpha)}$$

with  $L = h^2/k$ ,  $e = \frac{Ch^2}{k}$ , and  $C = \sqrt{A^2 + B^2}$ ,  $\tan \alpha = \frac{B}{A}$  as usual. Use **polarplot** in **Maple** to plot the orbits of Pluto ( $e = 0.2456$ ,  $L = 36.919$ ) and Neptune ( $e = 0.0110$ ,  $L = 30.008$ ) on the same polar graph. You may assume  $\alpha = 0$ .

\*\*If you have the “Applications Manual” that goes with our text, you may plot the orbits of any two (or more!) planets in our solar system, instead of (or in addition to!) Pluto and Neptune. There is a chart with the corresponding values of  $e$  and  $L$  on page 113. You may also use the internet or any texts from other classes that you may have to find these values for other planets.

## Exercises, Part 2:

You may be aware of the construction at Marriott the last few years. One objective of the construction is listed as “Improve seismic stability and bring the building up to code.” In short, the building has recently been “earthquake-proofed”, but what does this mean? What is the mathematics behind “the code” that Marriott failed to satisfy before? Why should we be more confident now that the library won’t collapse in an earthquake?

In this part of the project, you will study a non-homogeneous second order system which describes the oscillations of a 4-story building (all stories above ground) experiencing an earthquake. Suppose each of the 4 stories weighs 32 tons, so the mass of each is  $m = 2000$  slugs. Also, assume a horizontal restoring force of  $k = 5$  tons per foot (=10,000 lbs/ft) between the adjacent floors. Therefore, if  $x_i(t)$  describes the horizontal displacement of the  $i$ th floor, the forces exerted by the adjacent floors on the  $i$ th floor satisfy the differential equation (depending on  $i$ )

$$\begin{aligned} mx_1'' &= -kx_1 + k(x_2 - x_1) &= -2kx_1 &+ kx_2 \\ mx_2'' &= -k(x_2 - x_1) + k(x_3 - x_2) &= kx_1 &- 2kx_2 + kx_3 \\ mx_3'' &= -k(x_3 - x_2) + k(x_4 - x_3) &= kx_2 &- 2kx_3 + kx_4 \\ mx_4'' &= -k(x_4 - x_3) &= kx_3 &- kx_4 \end{aligned} \quad (6)$$

Here,  $x_1$  is the ground floor, with the  $-kx_1$  term corresponding to the restorative force exerted by the foundation of the building, and  $x_4$  is the top floor. Let  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  be the (horizontal) position vector of the floors. We can write the above system of second order equations as a single matrix equation of the form  $\mathbf{x}'' = \frac{k}{m} \mathbf{A} \mathbf{x}$ , where  $k\mathbf{A}$  is the coefficient matrix for the right hand side

of the equation. With the values given above for  $k$  and  $m$ , we have

$$\frac{k}{m}\mathbf{A} = \frac{10,000}{2000} \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -10 & 5 & 0 & 0 \\ 5 & -10 & 5 & 0 \\ 0 & 5 & -10 & 5 \\ 0 & 0 & 5 & -5 \end{bmatrix} \quad (7)$$

### Problems:

1. Use **Maple** to find the eigenvalues of the matrix  $\frac{k}{m}\mathbf{A}$ . Make a table (by hand or in **Maple**) like the one on page 444 of our text to display your four eigenvalues ( $\lambda$ ) and the corresponding frequencies ( $\omega = \sqrt{-\lambda}$ ) and periods ( $P = \frac{2\pi}{\omega}$ ). Arrange the table so the periods are listed in increasing order. These are the frequencies and periods of natural oscillations of the building.
2. Suppose an earthquake produces a horizontal oscillation of the ground, described by the function  $\frac{1}{4}\cos\omega t$ . Therefore, the inertial force exerted on each floor by the earthquake is  $\frac{m}{4}\omega^2\cos\omega t$ , and consequently, the motion of our building undergoing this earthquake is modelled by the nonhomogeneous system

$$\mathbf{x}'' = \frac{k}{m}\mathbf{A}\mathbf{x} + \left(\frac{1}{4}\omega^2\cos\omega t\right)\mathbf{b} \quad (8)$$

where  $\mathbf{b} = (1, 1, 1, 1)$ . Assume a particular solution vector takes the form  $\mathbf{x}_p(t) = \mathbf{c}\cos\omega t$  for some constant vector  $\mathbf{c}$ . Use the method of undetermined coefficients to determine  $\mathbf{c}$ , if the period ( $\frac{2\pi}{\omega}$ ) of the earthquake oscillations is 2 seconds. You will want to use **Maple** to solve for  $\mathbf{c}$  by setting up an equation similar to (32) on page 440 of the text, then solving that linear system using **Maple**, as you did in Maple Project 2. What happens if the period of the earthquake oscillations is 2.8 seconds?

3. For  $\omega$  arbitrary, again solve for  $\mathbf{c}$ , in terms of  $\omega$ . If we write the entries of  $\mathbf{c}$  as  $\mathbf{c} = (c_1, c_2, c_3, c_4)$ , let  $c := \sqrt{c_1^2 + c_2^2 + c_3^2 + c_4^2}$ . This will be a function of  $\omega$ . Plot  $c(\omega)$ . Observe the spikes in your plot correspond to the periods you found in problem 1. They indicate the vibrations of the earthquake are producing a type of resonance with respect to the natural modes of oscillation for your building. In these instances, the building is most susceptible to damage.