Name KEY

Student ID # ____________

Math 2250
Summer 2009
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EXAM II
Monday, July 27, 2009

Problem	Points	Score
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
	TOTAL	

(20 points) 1. Find the general solution to the homogeneous equation

$$y^{(4)} - 2y^{(3)} + 2y'' - 2y' + y = 0,$$

given $y = \cos x$ is one solution.

$\Rightarrow$ $y = \sin x$ is a solution

$$\begin{array}{r|l} & r^2 - 2r + 1 \\ \hline r^2+1 & r^4 - 2r^3 + 2r^2 - 2r + 1 \\ & -(r^4 \quad + r^2) \\ \hline & -2r^3 + r^2 - 2r + 1 \\ & -(-2r^3 \quad - 2r) \\ \hline & r^2 + 1 \end{array}$$

$r^2 - 2r + 1 = (r-1)^2$

$\therefore$ roots are $r = \pm i, 1$ (with multiplicity 2)

General solution:

$$\boxed{y = C_1 \cos x + C_2 \sin x + C_3 e^x + C_4 x e^x}$$

(20 points) 2. Find the particular solution y_p to the nonhomogeneous differential equation
$$y'' - y = x + e^x.$$

Undetermined Coefficients:

$y'' - y = 0$ has solutions e^x, e^{-x} $\hookrightarrow$ duplication.

$y_p = Ax + B + Cxe^x$

$y_p' = A + Ce^x + Cxe^x$

$y_p'' = 2Ce^x + Cxe^x$

$y_p'' - y_p = 2Ce^x - Ax - B = x + e^x$

$\Rightarrow$ $A = -1$

$B = 0$

$C = \frac{1}{2}$

$\therefore$ $\boxed{y_p = -x + \frac{1}{2}xe^x}$

Variation of Parameters

$y'' - y = 0$ has solutions $y_1 = e^x, y_2 = e^{-x}$

$W(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$, $f(x) = x + e^x$

$y_p = -\left(\int \frac{e^{-x}(x+e^x)}{-2}dx\right)e^x + \left(\int \frac{e^x(x+e^x)}{-2}dx\right)e^{-x}$

$= \frac{1}{2}\left(\int xe^{-x} + 1\, dx\right)e^x - \frac{1}{2}\left(\int xe^x + e^{2x}dx\right)e^{-x} = (*)$

$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$

$u = x$ $du = dx$

$dv = e^{-x}dx$ $v = -e^{-x}$

$\int xe^x dx = xe^x - e^x$

$\Rightarrow$ $(*) = \frac{1}{2}(-xe^{-x} - e^{-x} + x)e^x - \frac{1}{2}(xe^x - e^x + \frac{1}{2}e^{2x})e^{-x}$

$= -\frac{1}{2}x - \cancel{\frac{1}{2}} + \frac{1}{2}xe^x - \frac{1}{2}x + \cancel{\frac{1}{2}} - \frac{1}{4}e^x$

$= \boxed{-x + \frac{1}{2}xe^x - \frac{1}{4}e^x}$

(20 points) 3. Find the current $I(t)$ for the RLC circuit with $R = 4\Omega$, $L = 2H$, $C = 0.1F$, $E(t) = 20V$, $I(0) = 0$, and $Q(0) = 5$.

$$LI' + RI + \frac{1}{C}Q = E$$

$$\Rightarrow 2I'(0) + 4\cancel{I(0)} + 10Q(0) = E(0)$$

$$2I'(0) + 50 = 20 \quad \Rightarrow I'(0) = -15$$

$$\cancel{2}I'' + \cancel{4}^{2}I' + \cancel{10}^{5}\, I = 0$$

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$\Rightarrow I(t) = e^{-t}(A\cos(2t) + B\sin(2t))$$

$$I(0) = A = 0$$

$$\Rightarrow I(t) = e^{-t}(B\sin(2t))$$

$$I'(t) = -e^{-t}(B\sin 2t) + 2Be^{-t}\cos 2t$$

$$I'(0) = 2B = -15 \Rightarrow B = -\frac{15}{2}$$

$$\therefore \boxed{I(t) = -\frac{15}{2}e^{-t}\sin(2t).}$$

(20 points) 4. Apply the definition of Laplace transform to compute $\mathcal{L}\{\cosh t\}$.

$$\mathcal{L}\{\cosh t\} = \mathcal{L}\left\{\frac{e^t + e^{-t}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^t\} + \frac{1}{2}\mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{2}\int_0^\infty e^{-st}e^t\,dt + \frac{1}{2}\int_0^\infty e^{-st}e^{-t}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-1)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+1)t}\,dt$$

$$= \frac{1}{2}\left[-\frac{1}{s-1}e^{-(s-1)t}\Big|_0^\infty - \frac{1}{s+1}e^{-(s+1)t}\Big|_0^\infty\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+1}\right], \quad s > 1$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-1}\right] = \boxed{\frac{s}{s^2-1}, \quad s > 1}$$

(20 points) 5. Use Laplace transforms to solve the initial value problem

$$x'' - x = 1, \;\; x'(0) = x(0) = 0.$$

$$\mathcal{L}\{x''\} - \mathcal{L}\{x\} = \mathcal{L}\{1\}$$

$$s\mathcal{L}\{x'\} - \cancel{x'(0)}^{0} - \mathcal{L}\{x\} = \frac{1}{s}$$

$$s(s\mathcal{L}\{x\} - \cancel{x(0)}^{0}) - \mathcal{L}\{x\} = \frac{1}{s}$$

$$(s^2 \quad -1)\mathcal{L}\{x\} = \frac{1}{s}$$

$$\mathcal{L}\{x\} = \frac{1}{s(s^2-1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$\Rightarrow \quad 1 = A(s^2-1) + B(s^2-s) + C(s^2+s)$$

$$1 = (A+B+C)s^2 + (C-B)s - A$$

$$\Rightarrow \quad -A = 1 \qquad \Rightarrow \qquad A = -1$$

$$B = C \qquad\qquad 2B = 1 \Rightarrow B = \frac{1}{2} = C$$

$$A+B+C = 0$$

$$\therefore \quad \mathcal{L}\{x\} = -\frac{1}{s} + \frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right)$$

$$x(t) = -1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

$$x(t) = -1 + \frac{e^{t}+e^{-t}}{2}$$

$$\boxed{x(t) = -1 + \cosh t}$$

* Other methods were accepted!