Mathematics 1220 Exam III Review Solutions Fall 2006

- Taylor and Maclaurin Series (§10.8)

  1. $1+x-\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x^{2n}}{(2n)!}+\dfrac{x^{2n+1}}{(2n+1)!}\right)$ represents $f(x)=\sin x+\cos x$ for all x since $\sin x=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$ and $\cos x=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!}$ for all x.
  2. $e^2+e^2(x-2)+e^2\frac{(x-2)^2}{2!}+e^2\frac{(x-2)^3}{3!}+e^2\frac{(x-2)^4}{4!}$
  3. For $|x|<1$, $(1+x)^{1/2}=\sum_{n=0}^{\infty}\binom{1/2}{n}x^n$ and $(1-x)^{1/2}=\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n$, so $f(x)=2\sum_{n=0}^{\infty}\binom{1/2}{2n}x^{2n}$.
  4. $\dfrac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$ and $\dfrac{1}{x^2}=\sum_{n=1}^{\infty}(-1)^{n+1}n(x-1)^{n-1}=\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$ for all x in the interval $(0,2)$.
  5. $\ln x=\int_1^x\dfrac{1}{t}\,dt=\sum_{n=0}^{\infty}(-1)^n\dfrac{(x-1)^{n+1}}{n+1}$.

- Approximation by Taylor polynomials (§11.1)

  1. $g(x)=3+9(x-2)+8\frac{(x-2)^2}{2!}+6\frac{(x-2)^3}{3!}+R_3(x)$ where $R_3(x)=\frac{g^{(4)}(c)}{4!}(x-2)^4$, but $g^{(4)}(c)=0$ for all c, so $R_3(x)=0$.
  2. The fourth order Taylor approximation to $f(x)$ is $\frac{1}{2}-\frac{x-1}{2^2}+\frac{(x-1)^2}{2^3}-\frac{(x-1)^3}{2^4}+\frac{(x-1)^4}{2^5}$, so $f(1.2)$ is approximately 0.45455. The error term is $R_4(x)=-\frac{(x-1)^5}{(1+c)^6}$ for some c in $[1,1.2]$. $|R_4(1.2)|=|\frac{(0.2)^5}{(1+c)^6}|\leq|\frac{(0.2)^5}{2^6}|=0.000005$.
  3. $|R_n(0.3)|\leq\frac{(0.3)^{n+1}}{(n+1)!}\leq\frac{1}{(n+1)!}\leq 10^{-3}$ when $n\geq 6$. The sixth order Taylor approximation to $\cos x$ is $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$, so $\cos 0.3$ is approximately 0.9553364875.
  4. $\int_{0.8}^{1.2}\ln x\,dx=\int_{0.8}^{1.2}\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(x-1)^n}{n}\,dx=\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(x-1)^{n+1}}{(n+1)n}\Big|_{0.8}^{1.2}\approx(x-1)^2-\dfrac{(x-1)^3}{6}+\dfrac{(x-1)^4}{12}-\dfrac{(x-1)^5}{20}\Big|_{0.8}^{1.2}=-\dfrac{(0.2)^3}{3}-\dfrac{(0.2)^5}{10}=-0.002698\bar{6}$.

- Numerical Integration (§11.2)

  1. $\int_{0.8}^{1.2}\ln x\,dx\approx-0.002786$, $|E_8|=|\frac{(0.4)^3}{12(8)^2c^2}|\leq\frac{(0.4)^3}{12(8)^2(0.8)^2}=\frac{1}{7680}\approx 0.0001302$
  2. $\int_{0.8}^{1.2}\ln x\,dx\approx-0.002699$, $|E_8|=|\frac{(0.4)^5 3!}{180(8)^4c^4}|\leq\frac{(0.4)^5 3!}{180(8)^4(0.8)^4}\approx 2.0345\times 10^{-7}$.
  3. $|E_n|\leq\frac{1}{120n^2}\leq 10^{-3}$ if $n^2\geq 9$. Take $n=3$, then $\int_0^{0.5}e^{x^2}\,dx\approx 0.54795$

4. Let $f(x) = \frac{\sin x}{x}$. Then, $f^{(4)}(x) = \frac{(24-12x^2+x^4)\sin x+(-24x+4x^3)\cos x}{x^5} \leq \frac{1}{5}$ for x in $[0,1]$. Therefore, $|E_n| \leq \frac{1}{900n^4} \leq 10^{-5}$ if $n^4 \geq \frac{1000}{9}$, so we may take $n \geq 4$. The parabolic rule with $n = 4$ gives $\int_0^1 \frac{\sin x}{x}\, dx \approx 0.946087$

- Numerical Equation Solving (§11.2-11.3)

1. The root is approximately 0.60, accurate to 2 decimal places. It takes 6 iterations to achieve this accuracy.
2. $n \geq 40$
3. $\sqrt{2}$ is the positive root of the equation $x^2 - 2$, which lies in the interval $[1, 2]$. We will take $x_1 = 1.5$. Then, $x_4 = 1.4142135$ is accurate to five decimal places.
4. We will approximate the root in the interval $[2, 3]$ and take $x_1 = 1.2$. Then, $x_6 = 1.0893$ is accurate to 3 decimal places.
5. Newton's method isn't applicable because $x^{1/3}$ is not differentiable at 0.
6. The fixed point is 0.60, accurate to 2 decimal places.
7. $\tan x$ does not take $[4, 4.5]$ into itself, so we can not even guarantee a fixed point exists on this interval. Even if a fixed point exists in this interval, the derivative $\sec^2 x < 1$ if and only if $1 < \cos^2 x$ if and only if $\sin^2 x < 0$, which occurs for no x. Thus, the fixed point algorithm may not succeed.
8. Let $x_1 = 4.5$. Then, $x_3 = 4.4934$ approximates the fixed point accurately to 3 decimal places.

- Approximating solutions to differential equations (§11.5)

1. Below is a table of the results from Euler's method:

n	x_n	y_n
0	1	2
1	1.2	2.4
2	1.4	2.976
3	1.6	3.80928
4	1.8	5.0282496
5	2	6.838419456

2. Below is a table of the results from Improved Euler's method:

n	x_n	$\hat{y}_n$	y_n
0	1	2	2
1	1.2	2.4	2.488
2	1.4	3.08512	3.2184768
3	1.6	4.119650304	4.32820760064
4	1.8	5.71323403284	6.04910294265
5	2	8.22678000201	8.78329747273

3. Euler's method says $y_n = y_{n-1} + hy_{n-1} = y_{n-1}(1+h)$ for $n \geq 1$. Therefore, $y_n = y_{n-1}(1+h) = [y_{n-2}(1+h)](1+h) = y_{n-2}(1+h)^2$. By repeating this reduction a total of n times, we find

$$y_n = y_{n-1}(1+h) = y_{n-2}(1+h)^2 = \cdots = y_0(1+h)^n.$$

4. Below is a table of the results from Improved Euler's method:

n	x_n	$\hat{y}_n$	y_n
0	0	2	2
1	0.2	3.2	3.56
2	0.4	5.696	6.3368
3	0.6	10.13888	11.279504
4	0.8	18.0472064	20.07751712
5	1	32.124027392	35.7379804736
6	1.2	57.1807687578	63.613605243
7	1.4	101.781768389	113.232217333
8	1.6	181.171547732	201.553346853
9	1.8	322.485354964	358.764957398
10	2	574.023932837	638.601624168