Mathematics 1220 Exam II Review Solutions Fall 2006

- Indeterminate Forms (§9.1-9.2)

  (a) $\lim_{x\to\infty} \sqrt[x]{x} = \lim_{x\to\infty} e^{\ln x/x} = 1$, so $\lim_{x\to\infty} x\left(\sqrt[x]{x} - 1\right) = \lim_{x\to\infty} \dfrac{\sqrt[x]{x}-1}{1/x}$ is an indeterminate form of type $\dfrac{0}{0}$. Apply L'Hopital's rule to get $\lim_{x\to\infty} (\ln x - 1)\sqrt[x]{x} = \infty$.

  (b) $\lim_{x\to\infty} \dfrac{xe^{-x^2/2}}{e^{-x}} = \lim_{x\to\infty} \dfrac{xe^{x}}{e^{x^2/2}}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$. Apply L'Hopital's rule to get $\lim_{x\to\infty} \dfrac{e^x + xe^x}{xe^{x^2/2}} = \lim_{x\to\infty} \left(\dfrac{1}{xe^{x^2/2-x}} + \dfrac{1}{e^{x^2/2-x}}\right) = 0$.

  (c) $\lim_{x\to-\infty} (e^{-x} - x) = \lim_{x\to\infty} (e^x + x) = \infty$ is not an indeterminate form.

  (d) $\lim_{x\to 1^+} \dfrac{\int_1^x \sin t\, dt}{x-1}$ is an indeterminate form of type $\dfrac{0}{0}$. Apply L'Hopital's rule to get $\lim_{x\to 1^+} \sin x = \sin(1)$.

  (e) $\lim_{x\to\infty} \dfrac{3x}{\ln(100x + e^x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$. Apply L'Hopital's rule to get $\lim_{x\to\infty} 3\left(\dfrac{100x + e^x}{100 + e^x}\right)$, which is still an indeterminate form of type $\dfrac{\infty}{\infty}$. Apply L'Hopital's rule again to get $\lim_{x\to\infty} 3\left(\dfrac{100 + e^x}{e^x}\right) = \lim_{x\to\infty} 3\left(\dfrac{100}{e^x} + 1\right) = 3$.

  (f) $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2}$ is an indeterminate form of type $\dfrac{0}{0}$. Apply L'Hopital's rule twice to get $\lim_{x\to 0} \dfrac{-\sin x}{2x} = \lim_{x\to 0} \dfrac{-\cos x}{2} = \dfrac{1}{2}$.

- Improper Integrals (§9.3-9.4)

  (a) For the substitution $u = \ln x$, $\int_e^\infty \dfrac{dx}{x\sqrt{\ln x}} = \int_1^\infty \dfrac{1}{u^{1/2}} du = \infty$, therefore diverges.

  (b) $\ln x < x^{1/4}$ and $x^{3/2} < \sqrt{x^3 + 2x + 1}$ imply

  $$\frac{\ln x}{\sqrt{x^3+2x+1}} < \frac{x^{1/4}}{\sqrt{x^3+2x+1}} < \frac{x^{1/4}}{x^{3/2}} = \frac{1}{x^{5/4}}.$$

  Therefore, $\int_1^\infty \dfrac{\ln x}{\sqrt{x^3+2x+1}}\, dx \le \int_1^\infty \dfrac{1}{x^{5/4}}\, dx < \infty$, and so it converges.

(c) $\int_0^\infty e^{-x}\cos x\,dx = \left(e^{-x}\sin x - e^{-x}\cos x\right)|_0^\infty = 1$, therefore converges.

(d) $\int_{-\infty}^\infty \frac{e^{-x^2}}{x^2}\,dx = 2\int_0^\infty \frac{e^{-x^2}}{x^2}\,dx = 2\left(\int_0^1 \frac{e^{-x^2}}{x^2}\,dx + \int_1^\infty \frac{e^{-x^2}}{x^2}\,dx\right)$. For $x \geq 1$, $e^{-x^2} < \frac{1}{x}$ implies $\int_1^\infty \frac{e^{-x^2}}{x^2}\,dx \leq \int_1^\infty \frac{1}{x^3}\,dx < \infty$, but for $0 < x \leq 1$, $\frac{1}{e} \leq e^{-x^2}$ implies $\frac{1}{e}\int_0^1 \frac{1}{x^2}\,dx \leq \int_0^1 \frac{e^{-x^2}}{x^2}\,dx$, and $\int_0^1 \frac{1}{x^2}\,dx$ diverges, so $\int_{-\infty}^\infty \frac{e^{-x^2}}{x^2}\,dx$ diverges.

(e) $\int_0^\infty x^{16,000}e^{-x}\,dx = \int_0^1 x^{16,000}e^{-x}\,dx + \int_1^\infty x^{16,000}e^{-x}\,dx$. Since $\int_0^1 x^{16,000}e^{-x}\,dx$ is not an improper integral, we conclude immediately that it converges. On the other hand, for $x \geq 1$, $e^{-x} \leq x^{-16,002}$, so $\int_1^\infty x^{16,000}e^{-x}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx < \infty$, thus also converges.

(f) For $u = -\ln(\cos x)$, $\int_{\pi/3}^{\pi/2} \frac{\tan x}{(\ln\cos x)^2}\,dx = \int_{\ln 2}^\infty \frac{1}{u^2}\,du < \infty$, therefore converges.

- Infinite Sequences and Series (§10.1-10.2)

1. (a) $a_n = \frac{n!}{3^n} = n\frac{(n-1)!}{3^n}$. For $n \geq 4$, $(n-1)! \geq 2(3^{n-3})$, so $a_n \geq n\left(\frac{2}{27}\right)$. Therefore, a_n diverges.

   (b) $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2n}{2n+2} = 1$, but $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)\frac{2n+1}{2n+3} = -1$, therefore the sequence does not converge to a single number, so it diverges.

   (c) $a_n = \ln n - \ln(n+1) = \ln\left(\frac{n}{n+1}\right)$, so $\lim_{n\to\infty} a_n = \ln\left(\lim_{n\to\infty}\frac{n}{n+1}\right) = \ln 1 = 0$.

2. The recursion formula is $b_1 = 1$, $b_{n+1} = 1 + \frac{1}{b_n}$. If we say $b = \lim_{n\to\infty} b_n$, then b satisfies the equation $b = 1 + \frac{1}{b}$, which implies $b = \frac{1+\sqrt{5}}{2}$.

3. (a) $\sum_{n=0}^\infty e^{-2n} = \sum_{n=0}^\infty \left(\frac{1}{e^2}\right)^n$ is the geometric series with $r = e^{-2} < 1$, therefore converges to $\frac{1}{1-e^{-2}}$

   (b) $\sum_{k=1}^\infty \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \lim_{n\to\infty}\sum_{k=1}^n \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{k=1}^n \frac{1}{\sqrt{k+1}}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{k=2}^{n+1} \frac{1}{\sqrt{k}}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$, therefore converges to 1.

(c) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{-1/h} = e^{-1} \neq 0$. Therefore, by the nth term test, $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$ diverges.

(d) $\sum_{k=1}^{\infty}\frac{1}{16,000k} = \frac{1}{16,000}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, since $\sum_{k=1}^{\infty}\frac{1}{k}$ is the harmonic series.

4. We can describe the distance the bullfrog travels with the geometric series $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$. Since, $\frac{3}{4} < 1$, this series converges to $\frac{1}{1-\frac{3}{4}} = 4$, so the bullfrog travels 4 meters.

- Convergence of Infinite Series (§10.3-10.5)

(a) Let $a_n = \sqrt{1-\cos(\frac{1}{n})}$, $b_n = \frac{1}{n}$. Then, $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{1-\cos(1/n)}}{1/n} = \sqrt{\lim_{n\to\infty}\frac{1-\cos(1/n)}{1/n^2}} = \sqrt{\lim_{n\to\infty}\frac{\sin(1/n)}{1/n}} = 1$, so the limit comparison test implies $\sum_{n=1}^{\infty}\sqrt{1-\cos\left(\frac{1}{n}\right)}$ diverges, since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

(b) Let $a_n = \sqrt{n}\sin\left(\frac{1}{n}\right)$, $b_n = \frac{1}{\sqrt{n}}$. Then, $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n}\sin(1/n)}{1/\sqrt{n}} = \lim_{n\to\infty} n\sin(1/n) = 1$, so by the limit comparison test, the series diverges since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

(c) For the substitution $u = \ln x$, the integral $\int_2^{\infty}\frac{1}{x(\ln x)^{\pi}}\,dx = \int_{\ln 2}^{\infty}\frac{1}{u^{\pi}}\,du < \infty$, since $\pi > 1$. Therefore, by the integral test, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{\pi}}$ converges.

(d) $\lim_{n\to\infty}\frac{1}{n^{2/3}} = 0$, so by the alternating series test, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{2/3}}$ converges. Since $\frac{2}{3} < 1$, the series $\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$ diverges, so the alternating series does not converge absolutely.

(e) By the ratio test, $\lim_{n\to\infty}\frac{2^{n+1}(n+1)!}{(n+3)!}\frac{(n+2)!}{2^n n!} = \lim_{n\to\infty}\frac{2(n+1)}{(n+3)} = 2$ implies the series $\sum_{n=1}^{\infty}\frac{2^n n!}{(n+2)!}$ diverges.

(f) $\lim_{n\to\infty}\frac{n^{1/n}}{\ln n} = 0$ implies $\sum_{n=2}^{\infty}(-1)^n\frac{n^{1/n}}{\ln n}$ converges, but $n^{1/n} > 1$ and $n > \ln n$ for all $n \geq 2$ implies $\sum_{n=2}^{\infty}\frac{n^{1/n}}{\ln n} \geq \sum_{n=2}^{\infty}\frac{1}{\ln n} \geq \sum_{n=2}^{\infty}\frac{1}{n}$, which diverges. Therefore, by the ordinary comparison test, the series $\sum_{n=2}^{\infty}(-1)^n \dfrac{n^{1/n}}{\ln n}$ converges conditionally.

- Power Series, Taylor Series, Maclaurin Series (§10.6-10.8)

1. (a) $\lim_{n\to\infty} \dfrac{(n+1)!|x+1|^{n+1}}{3^{n+1}} \dfrac{3^n}{n!|x+1|^n} = \lim_{n\to\infty} \dfrac{(n+1)|x+1|}{3} < 1$ only when $x = -1$.

   (b) $\lim_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)^3+1} \dfrac{n^3+1}{|x|^n} = |x| < 1$, plus $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3+1}$ converges by the alternating series test and $\sum_{n=0}^{\infty} \frac{1}{n^3+1}$ and converges by the limit comparison test (with $b_n = \frac{1}{n^3}$), so the convergence set is the interval $[-1, 1]$.

   (c) $\lim_{n\to\infty} \dfrac{2^{n+2}|x|^{n+1}}{(2n+2)!} \dfrac{(2n)!}{2^{n+1}|x|^n} = \lim_{n\to\infty} \dfrac{2|x|}{(2n+2)(2n+1)} = 0 < 1$ for all x. Therefore, the convergence set is $(-\infty, \infty)$

2. (a) If $g(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, then $f(x) = \dfrac{1}{(1+x)^2} = -g'(-x) = \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

   (b) If $g(x) = tan^{-1}x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n^1}$, then $f(x) = \int_0^x \dfrac{\tan^{-1} t}{t}\, dt = \int_0^x \dfrac{g(t)}{t}\, dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)^2}$

   (c) If $g(x) = e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, then $f(x) = xe^{-x} = xg(-x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+1}}{n!}$

3. $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$, so $f(x) = \sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \ldots$. Since $|f^{(n+1)}(x)| = |\sin x + \cos x|$ or $|\sin x - \cos x|$ and each is at most 2, $|R_n(x)| \le \frac{2|x|^{n+1}}{(n+1)!}$ for all real numbers c, the Maclaurin series represents f for all real numbers x.

4. $e^2 \left(1 + (x-2) + \dfrac{(x-2)^2}{2!} + \dfrac{(x-2)^3}{3!} + \dfrac{(x-2)^4}{4!}\right)$