Mathematics 1220 Exam I Review Answers Fall 2006

- Transcendental Functions and their Properties (§7.1, 7.3-7.4, 7.7-7.8)

1. (a) $e^{x\ln x} = e^{\ln x^x} = x^x$

   (b) $e^{\ln 3 + 2\ln x} = e^{\ln 3 + \ln x^2} = e^{\ln 3x^2} = 3x^2$

   (c) $h = 3x \Rightarrow x = h/3$,
   $\lim_{x\to 0}(1+3x)^{1/x} = \lim_{h\to 0}(1+h)^{3/h} = \left(\lim_{h\to 0}(1+h)^{1/h}\right)^3 = e^3$

   (d) $h = -1/2n \Rightarrow n = -1/2h$,
   $\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^{6n} = \lim_{h\to 0}(1+h)^{-3/h} = \left(\lim_{h\to 0}(1+h)^{1/h}\right)^{-3} = e^{-3}$

   (e) $\sin(2\cos^{-1}x) = 2\sin(\cos^{-1}x)\cos(\cos^{-1}x) = 2x\sqrt{1-x^2}$

   (f) $\cos(\tan^{-1}x) = \frac{1}{\sec(\tan^{-1}x)} = \frac{1}{\sqrt{1+x^2}}$

2. (a) $\frac{dy}{dx} = \frac{3}{2(3x-2)}$

   (b) $\frac{dy}{dx} = \ln 7(2x+1)7^{(x+1)^2-x}$

   (c) $\frac{dy}{dx} = e^{\cos(x^2)} - 2x^2\sin(x^2)e^{\cos(x^2)}$

   (d) $\frac{dy}{dx} = \frac{-\sin x}{\sqrt{1-\cos^2 x}} = -1$

   (e) $\frac{dy}{dx} = \sec x\left(\tan x\tan^{-1}(e^x) + \frac{e^x}{1+e^{2x}}\right)$

   (f) $\frac{dy}{dx} = \frac{2\sec^2 x}{\tan x}$

   (g) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{x+1}} = \frac{1}{2\sqrt{x^2+x}}$

   (h) $\frac{dy}{dx} = e^x\text{sech}^2 x e^{\tanh(e^x)}$

   (i) $\frac{dy}{dx} = x^x(\ln x + 1)$

   (j) $\ln y = \frac{1}{3}\ln(x+3) + 2\ln(3x+2) - \frac{1}{2}\ln(x+1)$
   $\frac{1}{y}\frac{dy}{dx} = \frac{1}{3(x+3)} + \frac{6}{3x+2} - \frac{1}{2(x+1)}$
   $\frac{dy}{dx} = \frac{(x+3)^{1/3}(3x+2)^2}{\sqrt{x+1}}\left(\frac{1}{3(x+3)} + \frac{6}{3x+2} - \frac{1}{2(x+1)}\right)$

3. (a) $u = 2x^4 + 8x + 1 \Rightarrow du = 8(x^3+1)dx$,
   $\int \frac{x^3+1}{2x^4+8x+1}\,dx = \frac{1}{8}\int\frac{1}{u}\,du = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln|2x^4+8x+1| + C$

   (b) $u = \sin x \Rightarrow du = \cos x dx$, $\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C$

   (c) $u = \frac{-1}{x} \Rightarrow du = \frac{1}{x^2}dx$, $\int \frac{e^{-1/x}}{x^2}\,dx = \int e^u\,du = e^u + C = e^{-1/x} + C$

(d) $u = x^2 - 1 \Rightarrow du = 2xdx$,
$\displaystyle\int x10^{x^2-1}\,dx = \frac{1}{2}\int 10^u\,du = \frac{10^u}{2\ln 10} + C = \frac{10^{x^2-1}}{2\ln 10} + C$

(e) $u = 3x \Rightarrow du = 3dx$,
$\displaystyle\int \frac{1}{1+9x^2}\,dx = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \tfrac{1}{3}\tan^{-1}u + C = \tfrac{1}{3}\tan^{-1}3x + C$

(f) $u = x^2 \Rightarrow du = 2xdx$,
$\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\sin^{-1}(x^2) + C$

(g) $u = \text{tanh}x \Rightarrow du = \text{sech}^2 x dx$,
$\displaystyle\int \text{sech}^2 x\,\text{tanh}x\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\text{tanh}^2 x + C$

(h) $u = \text{cosh}x \Rightarrow du = \text{sinh}x dx$,
$\displaystyle\int \text{sinh}x\,\sinh(\text{cosh}x)\,dx = \int \text{sinh}u\,du = \text{cosh}u + C = \cosh(\text{cosh}u) + C$

- Inverse Functions and their Properties (§7.2)

1. (a) $f(x) = x^2 + x - 6 \Rightarrow f'(x) = 2x + 1$
$f'(x) > 0$ on $[-1/2, \infty)$, so $f^{-1}(x) = \sqrt{x + \frac{25}{4}} - \frac{1}{2}$ for $f(x)$ restricted to $[-1/2, \infty)$.
$f'(x) < 0$ on $(-\infty, -1/2]$, so $f^{-1}(x) = -\sqrt{x + \frac{25}{4}} - \frac{1}{2}$ for $f(x)$ restricted to $(-\infty, -1/2]$.

(b) $f(x) = \sqrt{\dfrac{1}{x-2}} \Rightarrow f'(x) = -\dfrac{1}{2}(x-2)^{-3/2}$
$f'(x) > 0$ on $(2, \infty)$ (note $f(x)$ is not defined elsewhere), so $f^{-1}(x) = \frac{1}{x^2} - 2$ for $f(x)$ restricted to $(2, \infty)$.

(c) $f(x) = \dfrac{\cos x - 1}{\cos x + 1} \Rightarrow f'(x) = \dfrac{-2\sin x}{(\cos x + 1)^2}$
$f'(x) > 0$ on $(\pi, 2\pi]$ (note that $f(x)$ is undefined at $x = \pi$), so $f^{-1} = \cos^{-1}\left(\frac{x+1}{x-1}\right) + \pi$ for $f(x)$ restricted to $(\pi, 2\pi]$ (recall that $\cos^{-1}(y) = x$ is defined for x in $[0, \pi]$, but our x are in $(\pi, 2\pi]$, so we can't just apply $\cos^{-1}$ to both sides).
$f'(x) < 0$ on $[0, \pi)$, so $f^{-1}(x) = \cos^{-1}\left(\frac{1+x}{1-x}\right)$ for $f(x)$ restricted to $[0, \pi)$.

2. (a) $(f^{-1})'(x) = \dfrac{1}{\sec^2(\tan^{-1}x)} = \dfrac{1}{1+x^2}$

(b) $f(x) = e^{x/3} \Rightarrow (f^{-1})'(x) = \frac{3x^2}{\frac{1}{3}e^{(\ln x^3)/3}} = 9x$

(c) $(f^{-1})'(x) = \frac{1}{-\text{csch}(\text{csch}^{-1}x)\text{coth}(\text{csch}^{-1}x)} = \frac{-1}{x\text{coth}(\text{csch}^{-1}x)}$

- Exponential Growth and Decay (§7.4)

1. The doubling period implies $2 = e^{2k}$, so $k = \ln 2/2$. You want to solve $1,000,000 = 10,000e^{(\ln 2/2)t}$ for $t =$ time in days. You find $t = \frac{2\ln 100}{\ln 2}$, which is approximately 13 days, so you will definitely not be sick the day of the exam.

2. For Stewart to achieve his goal, the annualized growth rate would have to satisfy the equation $1,000,000 = 5,000\left(1+\frac{r}{1}\right)^{10}$, so $r = (200)^{1/10} - 1$, which is approximately 70% (definitely not realistic!)

3. The general solution to the differential equation $\frac{d\theta}{dt} = -k(\theta - T)$ is $\theta(t) = Ce^{-kt} + T$. You know $T = 68$, $k = 0.5$, and $\theta(0) = 98$ (body temperature), which implies $C = 30$. Therefore, the particular solution pertaining to our scenario is $\theta(t) = 30e^{-t/2} + 68$. We want to find the time t in hours such that $\theta(t) = 85$. This t is $t = -2\ln\left(\frac{17}{30}\right)$, which is approximately 1.15 hours. That is, the person died approximately one hour and 9 minutes prior to 2 pm, so the time of death is 12:51 pm.

- First Order Linear Differential Equations (§7.6)

1. (a) Integrating factor: $\frac{1}{x}$, $y = x(e^x + C)$
   (b) Integrating factor: $\sin^2 x$, $y = \frac{1}{3}\sin x + C\csc^2 x$
   (c) Integrating factor: e^{x^2}, $y = \frac{1}{2} + Ce^{-x^2}$

2. The differential equation for this system is $\frac{dy}{dt} = 5 - \frac{y}{20}$, with initial condition $y(0) = 50$. We will solve the differential equation with the integrating factor $e^{t/20}$. The general solution is $y = 100 + Ce^{-t/20}$. The initial condition $y(0) = 50$ implies $C = -50$, so the particular solution is $y = 50(2 - e^{-t/20})$. The amount of time it takes for there to be 75 pounds of salt in the tank is $t = 20\ln 2$ (approximately 14) minutes.

- Techniques of Integration (§8.1-8.5)

1. (a) $\displaystyle\int \frac{\sin(2t)}{1-\sin^2(2t)}\,dt = \int \frac{\sin(2t)}{\cos^2(2t)}\,dt = \frac{1}{2}\int \frac{1}{u^2}\,du = \frac{-1}{2u} + C = \frac{-1}{2\cos(2t)} + C$
   (b) $\displaystyle\int \frac{\sec^2(\ln x)}{2x}\,dx = \frac{1}{2}\int \sec^2 u\,du = \tfrac{1}{2}\tan u + C = \tfrac{1}{2}\tan(\ln x) + C$
   (c) $\displaystyle\int \frac{xe^{\sqrt{2x^2+3}}}{\sqrt{2x^2+3}}\,dx = \frac{1}{2}\int e^u\,du = \tfrac{1}{2}e^u + C = \tfrac{1}{2}e^{\sqrt{2x^2+3}} + C$
2. (a) $\displaystyle\int \sin^{3/5}(2x+1)\cos^3(2x+1)\,dx = \tfrac{1}{2}\left(\tfrac{5}{8}\sin^{8/5}(2x+1) - \tfrac{5}{18}\sin^{18/5}(2x+1)\right) + C$
   (b) $\displaystyle\int \cos^4 x\,dx = \tfrac{3}{8}x + \tfrac{5}{16}\sin(2x) + C$
   (c) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)x\,dx = 0$
3. (a) $\displaystyle\int x^2(x+1)^{3/2}\,dx = \tfrac{2}{9}(x+1)^{9/2} - \tfrac{2}{3}(x+1)^3 + \tfrac{2}{5}(x+1)^{5/2} + C$
   (b) $\displaystyle\int \frac{2x-1}{x^2-6x+18}\,dx = 6\ln\left(\frac{3}{\sqrt{(x-3)^2+9}}\right) + 5\tan^{-1}\left(\frac{x-3}{3}\right) + C$
   (c) $\displaystyle\int \sqrt{5-4x-x^2}\,dx = \tfrac{9}{2}\sin^{-1}\left(\tfrac{x+2}{3}\right) + \tfrac{1}{2}(x+2)\sqrt{9-(x+2)^2} + C$