Mathematics 1220 Final Exam Review Answers Fall 2006

- Chapter 7: Transcendental Functions and First Order Differential Equations.

1. (a) $\dfrac{dy}{dx} = \ln 7(2x+1)7^{(x+1)^2-x}$ (b) $\dfrac{dy}{dx} = \dfrac{-\sin x}{\sqrt{1-\cos^2 x}} = -1$

   (c) $\dfrac{dy}{dx} = e^x \text{sech}^2 x e^{\tanh(e^x)}$ (d) $\dfrac{dy}{dx} = x^x(\ln x + 1)$

2. (a) $u = \sin x \Rightarrow du = \cos x dx$, $\displaystyle\int \cot x\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sin x| + C$

   (b) $u = x^2 - 1 \Rightarrow du = 2xdx$,
   $\displaystyle\int x10^{x^2-1}\, dx = \frac{1}{2}\int 10^u\, du = \frac{10^u}{2\ln 10} + C = \frac{10^{x^2-1}}{2\ln 10} + C$

   (c) $u = 3x \Rightarrow du = 3dx$,
   $\displaystyle\int \frac{1}{1+9x^2}\, dx = \frac{1}{3}\int \frac{1}{1+u^2}\, du = \tfrac{1}{3}\tan^{-1} u + C = \tfrac{1}{3}\tan^{-1} 3x + C$

   (d) $u = \tanh x \Rightarrow du = \text{sech}^2 x dx$,
   $\displaystyle\int \text{sech}^2 x \tanh x\, dx = \int u\, du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\tanh^2 x + C$

3. The doubling period implies $2 = e^{2k}$, so $k = \ln 2/2$. You want to solve $1,000,000 = 10,000e^{(\ln 2/2)t}$ for $t =$ time in days. You find $t = \frac{2\ln 100}{\ln 2}$, which is approximately 13 days, so you will definitely not be sick the day of the exam.

4. (a) Integrating factor: $\frac{1}{x}$, $y = x(e^x + C)$

   (b) Integrating factor: e^{x^2}, $y = \frac{1}{2} + Ce^{-x^2}$

- Chapter 8: Techniques of Integration

1. Trig techniques:
   $\displaystyle\int \sin^{3/5}(2x+1)\cos^3(2x+1)\, dx = \tfrac{1}{2}\left(\tfrac{5}{8}\sin^{8/5}(2x+1) - \tfrac{5}{18}\sin^{18/5}(2x+1)\right) + C$

2. Trig techniques: $\displaystyle\int \cos^4 x\, dx = \tfrac{3}{8}x + \tfrac{5}{16}\sin(2x) + C$

3. Rationalizing substitution:
   $\displaystyle\int \frac{2x-1}{x^2-6x+18}\, dx = 6\ln\left(\frac{3}{\sqrt{(x-3)^2+9}}\right) + 5\tan^{-1}\left(\frac{x-3}{3}\right) + C$

4. Rationalizing substitution:
   $\displaystyle\int \sqrt{5-4x-x^2}\, dx = \tfrac{9}{2}\sin^{-1}\left(\tfrac{x+2}{3}\right) + \tfrac{1}{2}(x+2)\sqrt{9-(x+2)^2} + C$

5. Integration by Parts: $\displaystyle\int \ln(x^2+4)\, dx = x\ln(x^2+4) - 2x + 4\tan^{-1}\left(\tfrac{x}{2}\right) + C$

6. Integration by Parts: $\displaystyle\int x\cot^2 x\, dx = x\cot x - \ln|\sin x| - \tfrac{1}{2}x^2 + C$

7. Partial Fractions: $\displaystyle\int \frac{x+9}{x^3+9x}\, dx = \ln x + \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) - \tfrac{1}{2}\ln|x^2+9| + C$

8. Partial Fractions: $\displaystyle\int \frac{x^4}{1-x^2}\, dx = -\tfrac{1}{3}x^3 - x + \tfrac{1}{2}\ln|1-x| + \tfrac{1}{2}\ln|1+x| + C$

9. Integration by Parts: $\int e^{x/3}\sin 3x\,dx = \frac{3}{82}e^{x/3}\sin 3x - \frac{27}{82}e^{x/3}\cos 3x + C$

- Chapter 9: Improper Integrals

1. $\ln x < x^{1/4}$ and $x^{3/2} < \sqrt{x^3+2x+1}$ imply

$$\frac{\ln x}{\sqrt{x^3+2x+1}} < \frac{x^{1/4}}{\sqrt{x^3+2x+1}} < \frac{x^{1/4}}{x^{3/2}} = \frac{1}{x^{5/4}}.$$

Therefore, $\int_1^\infty \frac{\ln x}{\sqrt{x^3+2x+1}}\,dx \le \int_1^\infty \frac{1}{x^{5/4}}\,dx < \infty$, and so it converges.

2. $\int_0^\infty e^{-x}\cos x\,dx = (e^{-x}\sin x - e^{-x}\cos x)\,|_0^\infty = 1$, therefore converges.

3. For $u = -\ln(\cos x)$, $\int_{\pi/3}^{\pi/2} \frac{\tan x}{(\ln\cos x)^2}\,dx = \int_{\ln 2}^\infty \frac{1}{u^2}\,du < \infty$, therefore converges.

- Chapter 10: Power Series Representations of Functions

1. (a) $\sum_{n=0}^\infty e^{-2n} = \sum_{n=0}^\infty \left(\frac{1}{e^2}\right)^n$ is the geometric series with $r = e^{-2} < 1$, therefore converges to $\frac{1}{1-e^{-2}}$

(b) $\sum_{k=1}^\infty \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \lim_{n\to\infty}\sum_{k=1}^n \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{k=1}^n \frac{1}{\sqrt{k+1}}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{k=2}^{n+1} \frac{1}{\sqrt{k}}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$, therefore converges to 1.

(c) $\sum_{k=1}^\infty \frac{1}{16,000k} = \frac{1}{16,000}\sum_{k=1}^\infty \frac{1}{k}$ diverges, since $\sum_{k=1}^\infty \frac{1}{k}$ is the harmonic series.

(d) For the substitution $u = \ln x$, the integral $\int_2^\infty \frac{1}{x(\ln x)^\pi}\,dx = \int_{\ln 2}^\infty \frac{1}{u^\pi}\,du < \infty$, since $\pi > 1$. Therefore, by the integral test, $\sum_{n=2}^\infty \frac{1}{n(\ln n)^\pi}$ converges.

(e) $\lim_{n\to\infty}\frac{1}{n^{2/3}} = 0$, so by the alternating series test, $\sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n^{2/3}}$ converges. Since $\frac{2}{3} < 1$, the series $\sum_{n=1}^\infty \frac{1}{n^{2/3}}$ diverges, so the alternating series does not converge absolutely.

(f) By the ratio test, $\lim_{n\to\infty}\frac{2^{n+1}(n+1)!}{(n+3)!}\frac{(n+2)!}{2^n n!} = \lim_{n\to\infty}\frac{2(n+1)}{(n+3)} = 2$ implies the series $\sum_{n=1}^\infty \frac{2^n n!}{(n+2)!}$ diverges.

2. (a) $\lim_{n\to\infty} \frac{(n+1)!|x+1|^{n+1}}{3^{n+1}} \frac{3^n}{n!|x+1|^n} = \lim_{n\to\infty} \frac{(n+1)|x+1|}{3} < 1$ only when $x = -1$.

   (b) $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)^3+1} \frac{n^3+1}{|x|^n} = |x| < 1$, plus $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3+1}$ converges by the alternating series test and $\sum_{n=0}^{\infty} \frac{1}{n^3+1}$ and converges by the limit comparison test (with $b_n = \frac{1}{n^3}$), so the convergence set is the interval $[-1, 1]$.

   (c) $\lim_{n\to\infty} \frac{2^{n+2}|x|^{n+1}}{(2n+2)!} \frac{(2n)!}{2^{n+1}|x|^n} = \lim_{n\to\infty} \frac{2|x|}{(2n+2)(2n+1)} = 0 < 1$ for all x. Therefore, the convergence set is $(-\infty, \infty)$

3. (a) If $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, then $f(x) = \frac{1}{(1+x)^2} = -g'(-x) = \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n$

   (b) If $g(x) = tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n^1}$, then $f(x) = \int_0^x \frac{\tan^{-1} t}{t}\, dt = \int_0^x \frac{g(t)}{t}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}$

   (c) If $g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $f(x) = xe^{-x} = xg(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$

4. $\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$ and $\frac{1}{x^2} = \sum_{n=1}^{\infty}(-1)^{n+1}n(x-1)^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$ for all x in the interval $(0, 2)$.

5. $\ln x = \int_1^x \frac{1}{t}\, dt = \sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{n+1}}{n+1}$.

- Chapter 11: Error in Numerical Techniques

  1. $|R_n(0.3)| \le \frac{(0.3)^{n+1}}{(n+1)!} \le \frac{1}{(n+1)!} \le 10^{-3}$ when $n \ge 6$. The sixth order Taylor approximation to $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$, so $\cos 0.3$ is approximately 0.9553364875.

  2. $|E_n| \le \frac{1}{120n^2} \le 10^{-3}$ if $n^2 \ge 9$. Take $n = 3$, then $\int_0^{0.5} e^{x^2}\, dx \approx 0.54795$

  3. Let $f(x) = \frac{\sin x}{x}$. Then, $f^{(4)}(x) = \frac{(24-12x^2+x^4)\sin x + (-24x+4x^3)\cos x}{x^5} \le \frac{1}{5}$ for x in $[0, 1]$. Therefore, $|E_n| \le \frac{1}{900n^4} \le 10^{-5}$ if $n^4 \ge \frac{1000}{9}$, so we may take $n \ge 4$. The parabolic rule with $n = 4$ gives $\int_0^1 \frac{\sin x}{x}\, dx \approx 0.946087$

  4. $n \ge 40$

  5. We will approximate the root in the interval $[2, 3]$ and take $x_1 = 1.2$. Then, $x_6 = 1.0893$ is accurate to 3 decimal places.

  6. Let $x_1 = 4.5$. Then, $x_3 = 4.4934$ approximates the fixed point accurately to 3 decimal places.

- Chapter 12: Conic Sections and Polar Coordinates

  1. (a) Parabola, focus $F = (1, 0)$, directrix $x = -1$
     (b) Ellipse, foci $F = (\pm\sqrt{5}, 0)$, directrices $x = \pm\frac{9\sqrt{5}}{5}$
     (c) Hyperbola, foci $F = (\pm 5\sqrt{5}, 0)$, directrices $x = \pm 4\sqrt{5}$
  2. (a) $\frac{(u-2)^2}{4} - \frac{v^2}{3} = 1$, angle of rotation $\theta = \frac{\pi}{4}$, center $(2, 0)$, hyperbola.
     (b) $\frac{u^2}{4} + \frac{v^2}{12} = 1$, angle of rotation $\theta = \frac{\pi}{4}$, center $(0, 0)$, ellipse.
     (c) $\frac{(u+2)^2}{5} + \frac{v^2}{25} = 1$, angle of rotation $\theta = \frac{\pi}{6}$, center $(-2, 0)$, ellipse.

- Chapter 18: Second Order Linear Differential Equations

  1. (a) $y = C_1 e^x + C_2 e^{-x}$ (b) $y = C_1 e^x + C_2 x e^x$ (c) $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$
  2. (a) $y_p = \frac{2}{3} \cos x + \frac{1}{6} \sin x$ (b) $y_p = \frac{1}{4} x e^{2x}$ (c) $y_p = \frac{1}{2} x^2 - x + 1$