Magic Squares

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Definition

A **magic square** is a $n \times n$ grid of numbers such that the sum of each row is equal, and equal to the sum of each column.

4	9	2
3	5	7
8	1	6

Some definitions also require the sum along the main diagonals to add to the same total.

A **perfect** magic square is a $n \times n$ square in which each of the entries $1, \ldots, n^2$ is used exactly once, and one in which the sum of the main diagonals is equal to the row (and column) sum.

Magic Squares: History

- There is a legend that the (semi-mythical) emperor Yu, c. 2200-2100 BCE, copied a magic square off the back of a giant turtle in the Luo, a tributary of the Huang He (Yellow River).
- The turtle's magic square is called the *Luo Shu* and is

4	9	2
3	5	7
8	1	6

- This story originated no later than 200 BCE.

Yang Hui's Constructions

The following method for constructing the Luo Shu and constructing a 4×4 magic square come from Yang Hui's book, 1275 CE:

"Xu Gu Zhai Suan Fa"
"Continuation of Ancient Mathematical Methods for Elucidating the Strange Properties of Numbers"

Method for Constructing the Luo Shu (c. 1275)

- Arrange numbers so that they slant downward, to the right

		1		
	4		2	
7		5		3
	8		6	
		9		

Method for Constructing the Luo Shu (c. 1275)

- Arrange numbers so that they slant downward, to the right
- Interchange the top and the bottom (1 and 9)

		1		
	4		2	
7		5		3
	8		6	
		9		

Method for Constructing the Luo Shu (c. 1275)

- Arrange numbers so that they slant downward, to the right
- Interchange the top and the bottom (1 and 9)
- Interchange the left and rightmost entries (7 and 3)

- 1
4 2
7 5 3
8 6
9
- 9
4 2
3 5 7
8 6
1

Method for Constructing the Luo Shu (c. 1275)

- Arrange numbers so that they slant downward, to the right
- Interchange the top and the bottom (1 and 9)
- Interchange the left and rightmost entries (7 and 3)
- Lower 9 to fill the slot between 4 and 2, raise 2 to fill the slot between 8 and 6

		1		
	4		2	
7		5		3
	8		6	
		9		

		9		
	4		2	
3		5		7
	8		6	
		1		

4	9	2
3	5	7
8	1	6

Yang Hui's Method of Constructing 4×4 Magic Squares

- Arrange the numbers 1 to 16 in order in a 4×4 array

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Yang Hui's Method of Constructing 4×4 Magic Squares

- Arrange the numbers 1 to 16 in order in a 4×4 array
- Interchange the numbers in the corner of the outer square

-

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

-

16	2	3	13
5	6	7	8
9	10	11	12
4	14	15	1

Yang Hui's Method of Constructing 4 × 4 Magic Squares

- Arrange the numbers 1 to 16 in order in a 4 × 4 array
- Interchange the numbers in the corner of the outer square
- Interchange the numbers at the corners of the inner square

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	2	3	13
5	6	7	8
9	10	11	12
4	14	15	1

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Neat Properties of Yang Hui's 4 × 4 square

Note that the sum of each quadrant is 34, the same as the row/column sum, as is the sum of the four outer corners, and the center square.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

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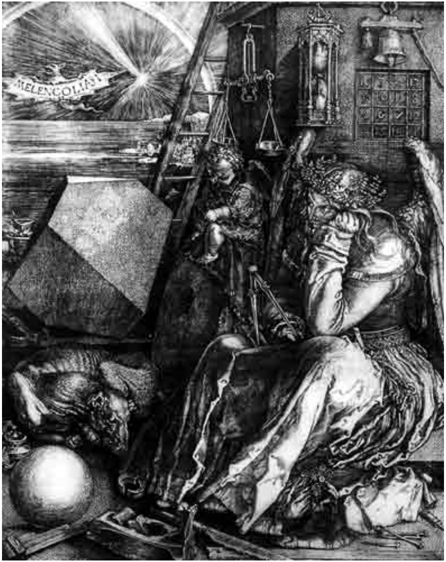
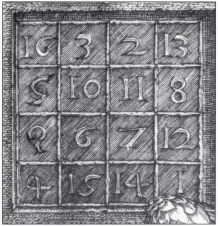
16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Quick Tangent: Magic Square in Durer's Melancholia (1514)



- Includes a 4×4 magic square!
- Etched in 1514

Durer's Melancholia (1514) and the DaVinci Code



"Exactly," Langdon said. "But did you know that this magic square is famous because Dürer accomplished the seemingly impossible?" He quickly showed Katherine that in addition to making the rows, columns, and diagonals add up to thirty-four, Dürer had also found a way to make the four quadrants, the four center squares, and even the four corner squares add up to that number. "Most amazing, though, was Dürer's ability to position the numbers 15 and 14 together in the bottom row as an indication of the year in which he accomplished this incredible feat!"
Katherine scanned the numbers, amazed by all the combinations.

- Dan Brown, *the DaVinci Code*

Yang Hui's 7×7 magic square

46	8	16	20	29	7	49
3	40	12	14	18	41	47
44	37	33	23	19	13	6
28	15	11	25	29	35	22
5	24	31	27	17	26	45
48	9	38	36	32	10	2
1	42	34	30	21	43	4

Contains a 5×5 magic square and a 3×3 magic square!
Yang Hui did not write how he found it!

Counting Perfect Magic Squares

There are no perfect 2×2 magic squares:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 2 \\ \hline \end{array}$$

Counting Perfect Magic Squares

There are no perfect 2×2 magic squares:

1	2
3	4

,

1	4
3	2

Once I've fixed one entry, I need two entries to be non-distinct:

1	

1	$\bigstar$
$\bigstar$	

3×3 perfect magic squares

- Suppose we want a 3×3 perfect magic square, i.e, using the digits $1, 2, \ldots, 9$.

$$1 + 2 + 3 + \cdots + 9 = (9)(10)/2 = 45$$

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- thus row sum (and column sum) $= 45/3 = 15$

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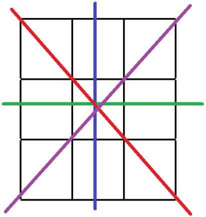
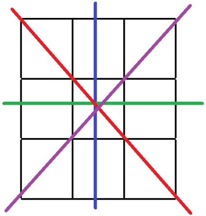
$$1 + 2 + 3 + \cdots + 9 = (9)(10)/2 = 45$$

- thus row sum (and column sum) $= 45/3 = 15$
- note that whatever number is in the middle needs to be part of 4 different ways to add up to 15



3×3 perfect magic squares

Row (and column) sum is 15



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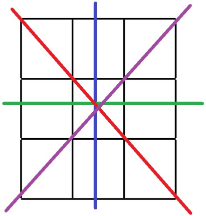

- $45 + 3c = 60$ (every entry, overcounting the center square 3 times, gives me 4×15)

3×3 magic squares

- Row (and column) sum is 15
- Center entry must be 5

	5	

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- whatever numbers go in a diagonal spot must be in 3 different partitions of 15
  - 1 can't be in a diagonal:
    $1+(6+8)= 15$,
    $1+(5+9)=15$
  -

	9	
	5	
	1	

3 × 3 magic squares

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	5	

- whatever numbers go in a diagonal spot must be in 3 different partitions of 15

  - 1 can't be in a diagonal: $1+(6+8)= 15$, $1+(5+9)=15$
  - 3 can't be in a diagonal: $3+(7+5)$, $3+(8+4)$

	9	
	5	
	1	

	9	
7	5	3
	1	

3×3 magic squares

We have:

	9	
7	5	3
	1	

- We know 6 and 8 can't go in the top row (9+6=15, 9+8=17)

3 × 3 magic squares

We have:

	9	
7	5	3
	1	

- We know 6 and 8 can't go in the top row (9+6=15, 9+8=17)
- We need to put 2 and 4 in top row, but

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	1	

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	1	

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- So we must have:

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This is it, up to rotations and reflections!

Number of Perfect Magic Squares

If we want to consider "perfect" magic squares, i.e. those with numbers 1 through n^2 whose main diagonals also add up to the row sum, there are

- 0 squares of size 2×2
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- 880 such squares of size 4×4 (up to rotations and reflections). This was shown by Frenicle, before 1675.
- 275,305,224 of size 5×5 (up to rotations and reflections)
- The number of perfect magic squares of size 6×6 is unknown! Estimated to be near $1.7745 \cdot 10^19$.

Counting (Not perfect) Magic Squares

- Suppose we want to count the number of $n \times n$ magic squares with row sum r.
- Let $H_n(r)$ denote this number
- Some gimmes:
  - $H_n(0) = 1$
  - $H_1(r) = 1$

Number of Size 2 Magic Squares With Any r

- To construct a size 2 magic square with row sum r, we can put any integer, i, from 0 to r in the first corner.
-

i	

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- The numbers in the same row and column as i are now fixed.

i	

i	r-i
r-i	

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-
i	

-
i	r-i
r-i	

-
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i	

i	r-i
r-i	

i	r-i
r-i	i

We got to make 1 choice, and we had $r+1$ options, so

$$H_2(r) = r + 1$$

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To create a magic square with row sum 1, in the first row, we have n options:

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- $$\begin{bmatrix} 0 & 1 & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}$$

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- For the second row, we have $n - 1$ options:
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & 0 & 0 \end{bmatrix}$$
- Until finally, the n^{th} row has only one option:
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$
giving a total of
$$H_n(1) = n!$$

Birkhoff-von Neumann

- Size $n \times n$ matrices of 0's and 1's, with exactly one 1 in each row and each column are called **permutation matrices**.
- There are $n!$ permutation matrices of size $n \times n$.

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- Size $n \times n$ matrices of 0's and 1's, with exactly one 1 in each row and each column are called **permutation matrices**.
- There are $n!$ permutation matrices of size $n \times n$.
- Note: If A and B are magic squares of size $n \times n$, then so is $A + B$.
- **Birkhoff-von Neumann Theorem**
  Every $n \times n$ magic square with row sum r is the sum of r (not necessarily distinct) permutation matrices of size $n \times n$.

Birkhoff-von Neumann for Counting 3×3 magic squares

- We know there are $3! = 6$ permutation matrices of size 3×3.
- To construct a size 3 magic square with row sum r, we will be choosing r objects from these 6 possibilities

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Lemma

The way to choose r, not necessarily distinct, objects from 6 options is $\binom{r+5}{r}$.

Number of ways to choose r not necessarily distinct objects from 6 options

Suppose we know we have r matrices, each of which can be one of 6 options.
Why isn't the answer r^6? Because order doesn't matter:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Instead, we can count the number of options by using **stars and bars**

Stars and Bars: part 1 of 2

Suppose we have r stars (for my pictures, $r = 8$)

$$\star\star\star\star\star\star\star\star$$

each of which can be one of 6 options. Call the options

$$P_1, P_2, P_3, P_4, P_5 \text{ and } P_6$$

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We are placing each of our r stars into one of 6 bins; which we can do by inserting 5 “bars” to represent the edges of the bins:

$$\star\star|\star|\star|\star|\star|\star\star$$

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$$\star\star|\star|\star|\star|\star|\star\star$$

corresponds to having

$$2P_1 + P_2 + P_3 + P_4 + P_5 + 2P_6$$

and $\star\star\star||\star|\star|\star|\star\star$

Stars and Bars: part 1 of 2

Suppose we have r stars (for my pictures, $r = 8$)

★★★★★★★★

each of which can be one of 6 options. Call the options

$$P_1, P_2, P_3, P_4, P_5 \text{ and } P_6$$

We are placing each of our r stars into one of 6 bins; which we can do by inserting 5 "bars" to represent the edges of the bins:

★★|★|★|★|★|★★

corresponds to having

$$2P_1 + P_2 + P_3 + P_4 + P_5 + 2P_6$$

and ★★★||★|★|★|★★ corresponds to

$$3P_1 + \quad P_3 + P_4 + P_5 + 2P_6$$

Stars and Bars: Part 2 of 2

In other words, we have $r + 5$ objects, r stars and 5 bars.

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

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$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Choosing the position of the r stars leaves exactly 5 positions left for the 5 bars, so there are $\binom{r+5}{r}$

$$\bigstar\bigstar\bigstar\_\_\_\bigstar\_\bigstar\_\bigstar\_\bigstar\bigstar$$

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$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

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$\star\star\star\_\_\_\star\_\star\_\star\_\star\star$

$\star\star\star||\star|\star|\star|\star\star$

A problem we didn't count on...

So there are $\binom{r+5}{r}$ ways to have a sum of r permutation matrices of size 3×3.

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Unfortunately...

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In other words, we have a ...

SYZYGY!!!

Subtracting off Syzygies

To count the number of 3×3 magic squares, we have an upper bound of $\binom{r+5}{r}$.
Every time we see $P_4 + P_5 + P_6$ I can replace it with $P_1 + P_2 + P_3$.

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Every time we see $P_4 + P_5 + P_6$ I can replace it with $P_1 + P_2 + P_3$.
I've overcounted everything with a $P_4 + P_5 + P_6$ in it
The number of ways to choose r many things, where three of them are fixed, is the number of ways to choose $r - 3$ many things:

$$\binom{(r-3)+5}{r-3}$$

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The number of ways to choose r many things, where three of them are fixed, is the number of ways to choose $r - 3$ many things:

$$\binom{(r-3)+5}{r-3}$$

So the number of 3×3 magic squares is

$$\binom{r+5}{r} - \binom{r+2}{r-3}$$

There are no other syzygies in this case, so we are done!
Hilbert's Syzygy Theorem promised us the process would terminate after $n!$ steps.

Syzygetic Method for 4×4 magic squares (R.P. Stanley)

Number of 4×4 magic squares:

$$\binom{r+9}{9} + 14\binom{r+8}{9} + 87\binom{r+7}{9} + 148\binom{r+6}{9} + 87\binom{r+5}{9} + 14\binom{r+4}{9} + \binom{r+3}{9}$$

About how many magic squares are there, though?

Every $H_n(r)$ is a polynomial in r of degree $(n-1)^2$
(Conjectured by Anand-Dumir-Gupta in 1966, proven by R.P. Stanley in 1973)

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- $H_1(r) = 1$ is a polynomial in degree 0
- $H_2(r) = r + 1$ is a polynomial in degree 1
- $$H_3(r) = \binom{r+5}{r} - \binom{r+2}{r-3} = \frac{1}{8}(r+1)(r+2)(r^2+3r+4)$$

Thank You!