The study of local cohomology was invented to answer a question about Unique Factorization Domains.

Size of a Ring
The last nonzero local cohomology of a ring measures how “big” the ring is.
An ideal \( I \) is a subset of a commutative ring \( R \):
\( I \) is an ideal of \( R \) if and only if:
\( \{0\} \subset I \subseteq R \) is an ideal of \( \mathbb{Z} \),
\( \{0\} \subset \mathbb{Z} \) is a ring,
\( \{0\} \subset \mathbb{Z} \) is a ring.

Examples of Ideals
• the set \( \{1\} \) is an ideal for all rings,
• the set \( \{0\} \) is an ideal for all rings,
• the set \( \{0\} \) is an ideal of \( \mathbb{Z} \),
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• the set \( \{0\} \) is an ideal of \( \mathbb{Z} \).

Local Cohomology Measures Size
The local cohomologies of a ring are a sequence that starts counting at 0 and tells you something about the ring.
- The 0th local cohomology of \( \mathbb{Z} \) is \( \{0\} \).
- The 1st local cohomology of \( \mathbb{Z} \) is \( \{0\} \).
- The 2nd local cohomology of \( \mathbb{Z} \) is \( \{0\} \).

Examples of Dimension
• the \( \mathbb{Z} \) is the only prime ideal in \( \mathbb{Z} \).
• \( \mathbb{Z} \) has two prime ideals:

A weird ring
Consider the ring of polynomials where we can use \( x, y \) as variables, and \( x, y \) as coefficients.
Everything in this ring is a polynomial with \( x \) and \( y \) (that is, \( \mathbb{Z}[x, y] \)), but this weird ring doesn’t have \( x, y, x^2y^2, \) etc. in it.

Local Cohomology Measures Weirdness
How far the first nonzero local cohomology is from the last nonzero local cohomology measures the weirdness of the ring.

Examples of Quotient Rings
• If \( I = (-\cdots - 7, 0, 7, 14, 21, \ldots) \subset \mathbb{Z} \), then
\[ I^2 = (-\cdots - 49, 0, 49, 98, 147, \ldots) \]• If \( I = \{\text{all polynomials with no constant term}\} \subset \mathbb{Q}[x] \), then
\[ I^2 = \{\text{all polynomials with no constant term and no constant term}\} \]

My Problem
Consider the ring with rational coefficients and 6 variables, arranged into a matrix:
\[
\begin{bmatrix}
1 & 2 & 3 & x & y & z
\end{bmatrix}
\]
Now consider the ideal \( I = \langle xz - wy, wz - vx, wz - yz \rangle \), that is, the ideal generated by \( x, y, z, x, y, z \) minors of the matrix.
By Hochster and Eagon, the quotient ring
\[
\frac{\mathbb{Q}(x, y, z)}{I}
\]
is Cohen-Macaulay! However, the ring \( \mathbb{R}/I, \) where \( I = \langle x, y \rangle \), is not Cohen-Macaulay.
My project centers around understanding the other local cohomology modules.

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Examples of Rings
• The integers, \( \mathbb{Z} \), –1, –2, 1, 2, 3, …
• The rational numbers, \( \mathbb{Q} \), \( 1/2, 1/3, \ldots \)
• Rational polynomials, \( \mathbb{Q}[x] \), \( 1 + 2x, 1/2 - 17x^2 - 2x^4 5\)
• Integers, \( \mathbb{Z} \), AND the square root of 5 are all numbers that look like:

For example,
\[ 1 + \sqrt{5}, \ 2\sqrt{5}, \ 3 - 6\sqrt{5} \]
We can add, subtract, and multiply:
\[(1 + \sqrt{5})(2\sqrt{5}) = 2\sqrt{5} + (1 + \sqrt{5})\sqrt{5} = \]
This ring is referred to as \( \mathbb{Z}[\sqrt{5}] \).
• The set of all \( n \times n \) matrices is a non commutative ring

Unique Factorization Domains
In the integers \( \mathbb{Z} \) a number can be factored into a set of prime (only divisable by itself and 1) numbers.
\[ \pm 2 = 2 \times 2 \times 3 \]
We can rearrage, but we can’t choose different primes.
In \( \mathbb{Z}[\sqrt{5}] \), what is and isn’t a prime is not so clear... It turns out 1 is prime
\( \mathbb{Z}[\sqrt{5}] \) is prime in \( \mathbb{Z}[\sqrt{5}] \) since it is and isn’t a prime and 2, but then
\[(1 + \sqrt{5})(1 - \sqrt{5}) = 6 = 2 \times 3 \]
Rings like \( \mathbb{Z} \), where numbers can be uniquely factored, are called Unique Factorization Domains.
Rings like \( \mathbb{Z}[\sqrt{5}] \) are not Unique Factorization Domains.

Examples of Subsets
• the set \( \{0, 1, 2, 3, \ldots\} \) is the ideal generated by 7
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Examples of Quotient Rings
• The ring \( \mathbb{Z}/(7) \) is the set of representatives \( \{0, 1, 2, 3, 4, 5, 6\} \) such that the product of two integers is their remainder when divided by 7. For example, \( 3 \times 3 = 9 \) in \( \mathbb{Z}/7 \).
• The ring \( \mathbb{Q}[x, y]/(x) \) is isomorphic to \( \mathbb{Q}[y] \), as we have essentially declared that \( x = 0 \)

The product of two ideals, \( I, J \) in \( R \) is the ideal \( IJ \) generated by all products \( x_iy_j \) where \( x_i \in I, y_j \in J \).
Then the power of an ideal \( I \) is the ideal, \( I^n \), generated by all products \( x_1^ny_1^m \) such \( x_i \in I \), \( n \in \mathbb{N} \).