Commutative Algebra and Local Cohomology

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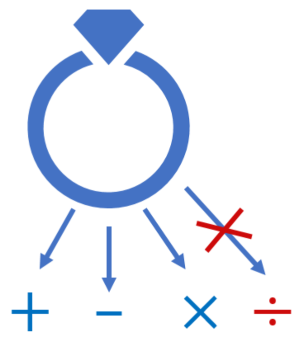
Set Up: Rings

A *ring* R is a set together with two binary operations $+$ and $\times$ (called addition and multiplication) satisfying the following axioms:

- $(R, +)$ is an abelian group
- $(a \times b) \times c = a \times (b \times c)$
- $(a + b) \times c = (a \times c) + (b \times c)$
- $a \times (b + c) = (a \times b) + (a \times c)$
- If $a \times b = b \times a$ for all elements, then R is called a *commutative ring*



Examples of Rings

- The integers, $\mathbb{Z}$: $-3, -2, -1, 0, 1, 2, 3, \ldots$
- The rational numbers, $\mathbb{Q}$: $1, \frac{3}{4}, \frac{17}{266}, \ldots$
- Rational polynomials, $\mathbb{Q}[x]$: $1, 7 + 2x$, $\frac{3}{4}x^2 - 17x^{10} - \frac{2}{3}x^{11}$
- Integers, $\mathbb{Z}$, *AND* the square root of -5 are all numbers that look like:

  $$\text{(integer)} + \text{(integer)}\sqrt{-5}:$$

  For example,

  $$1 + \sqrt{-5}, \quad 2\sqrt{-5} \quad , 3 - 6\sqrt{-5}$$

  We can still add, subtract, and multiply:

  $$(1 + \sqrt{-5})(2\sqrt{-5}) =$$
  $$2\sqrt{-5} + 2(\sqrt{-5})(\sqrt{-5}) =$$
  $$-10 + 2\sqrt{-5}$$

  This ring is referred to as $\mathbb{Z}[\sqrt{-5}]$
- The set of all $n \times n$ matrices is a *non commutative* ring

Unique Factorization Domains

In the integers ($\mathbb{Z}$) a number can be factored into a set of prime (only divisible by itself and 1) numbers.

$$12 = 2 \times 2 \times 3$$

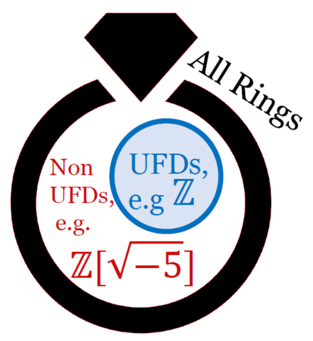
We can rearrange, but we can't choose different primes.
In $\mathbb{Z}[\sqrt{-5}]$, what *is* and *isn't* a prime is not so clear... It turns out

$$1 + \sqrt{-5}$$

is **prime** in $\mathbb{Z}[\sqrt{-5}]$ So are 3 and 2, but then:

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \times 3$$

Rings like $\mathbb{Z}$, where numbers can be uniquely factored, are called **Unique Factorization Domains**.
Rings like $\mathbb{Z}[\sqrt{-5}]$ are *not* **Unique Factorization Domains** .



The study of **local cohomology** was invented to answer a question about Unique Factorization Domains.

Size of a Ring

The last nonzero local cohomology of a ring measures how "big" the ring is.

An *ideal* I is a subset of a commutative ring R

- $(I, +)$ is an abelian group (closed under addition)
- For all $r \in R, a \in I$, $ra \in I$ (closed under multiplication)

Examples of Ideals

- the set $\{0\}$ is an ideal for all rings
- the ring is, itself, an ideal for all rings
- the set $(\ldots, -7, 0, 7, 14, 21, \ldots)$ is an ideal of $\mathbb{Z}$
- the set of all polynomials in x with no constant term,

  $$\{a_1x + a_2x^2 + \ldots a_nx^n | a_i \in \mathbb{Q}\}$$

  is an ideal of $\mathbb{Q}[x]$
- the set of all polynomials in x and y with no constant term,

  $$\{a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + \cdots + a_{n,m}x^ny^m\}$$

  is an ideal of $\mathbb{Q}[x, y]$
- the set of all polynomials in x and y such that every term has at least one x is an ideal of $\mathbb{Q}[x, y]$

Let A be some subset of a ring. The *ideal generated by* A is the smallest ideal containing A.
The ideal generated by A can also be thought of as the set of all finite sums of elements of the form ra, where $r \in R$ and $a \in A$.

Examples of Ideals Generated by Subsets

- the set $\{\ldots, -7, 0, 7, 14, \ldots\}$ is the ideal generated by 7
- the subset of $\mathbb{Q}[x]$ that is all polynomials in x with no constant term is the ideal generated by x
- the subset of $\mathbb{Q}[x, y]$ that is all polynomials such that every term has at least one x is the ideal generated by x in $\mathbb{Q}[x, y]$
- the subset of $\mathbb{Q}[x, y]$ that is all polynomials with no constant term is the ideal generated by x and y in $\mathbb{Q}[x, y]$

An *prime* ideal I is an ideal such that

- $I \neq R$
- if an element, $ab \in I$, then $a \in I$ or $b \in I$.

The *dimension* of a ring is the *longest chain of distinct prime ideals:*

$$I_0 \subset I_1 \subset \cdots \subset I_n$$

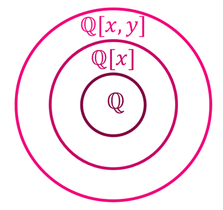
Examples of Dimension

- $\dim(\mathbb{Q}) = 0$:

  $$\{0\} \text{ is the only prime ideal in } \mathbb{Q}$$
- $\dim(\mathbb{Q}[x]) = 1$:

  $$\{0\} \subsetneq \{a_1x + a_2x^2 + \ldots a_nx^n | a_i \in \mathbb{Q}\}$$
- $\dim(\mathbb{Q}[x, y]) = 2$:

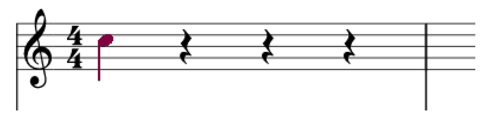
  $$\{0\} \subsetneq (x) \subsetneq (x, y)$$



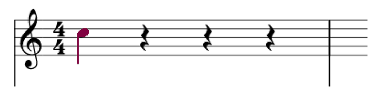
Local Cohomology Measures Size

The **local cohomologies** of a ring are a **sequence** that starts counting at 0 and tells you something about the ring.

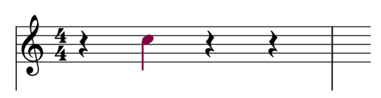
- The 0^{th} local cohomology of $\mathbb{Q}$ is $\mathbb{Q}$
- The 1^{st} local cohomology of $\mathbb{Q}$ is 0
- The 2^{nd} local cohomology of $\mathbb{Q}$ is 0



- The last nonzero local cohomology of $\mathbb{Q}$ is the 0^{th} one



- The last nonzero local cohomology of $\mathbb{Q}[x]$ is the 1^{st} one



- The last nonzero local cohomology of $\mathbb{Q}[x, y]$ is the 2^{nd} one

A weird ring

Consider the ring of polynomials where we can use x^4,x^3y,xy^3, and y^4 as variables, and rational numbers as coefficients, e.g.

$$\frac{1}{2}x^4 + x^3y + xy^3, \quad x^7y \quad x^4y^4$$

Everything in this ring is a polynomial with x and y (that is, $\mathbb{Q}[x, y]$); but this weird ring doesn't have x, y, x^2y^2, etc. in it.

Local Cohomology Measures Weirdness

How far the first nonzero local cohomology is from the last nonzero local cohomology measures the weirdness of the ring.

$\mathbb{Q}[x, y]$ has one nonzero local cohomology

$\mathbb{Q}[x^4, x^3y, xy^3, y^4]$ has two nonzero local cohomologies

Rings with only one nonzero local cohomology are called **Cohen Macaulay Rings**

Quotient Rings and Powers of Ideals

The quotient group R/I inherits a unique multiplication from the ring, R, which makes R/I itself a ring.

Examples of Quotient Rings

- The ring $\mathbb{Z}/(7)$ is the set of representatives $\{0, 1, 2, 3, 4, 5, 6\}$ such that the product of two integers is their remainder when divided by 7. For example, $2\dot{5} = 3$ in $\mathbb{Z}\,(7)$
- The ring $\mathbb{Q}[x, y]/(x)$ is isomorphic to $\mathbb{Q}[y]$, as we have essentially declared that (x) is 0

Note that quotienting out by an ideal can change the local cohomology quite a bit!

The product of two ideals, I, J in R is the ideal IJ generated by all products xy where $x \in I, y \in J$.
Then the power of an ideal I is the ideal, I^t, generated by all products $x_1 \ldots x_t$ where $x_i \in I$.

Examples of Powers of Ideals

- If $I = \{\cdots - 7, 0, 7, 14, 21, \ldots\} \subset \mathbb{Z}$, then

  $$I^2 = \{\cdots - 49, 0, 49, 98, 147, \ldots\}$$
- If $I = \{\text{all polynomials with no constant term }\} \subseteq \mathbb{Q}[x]$, then

  $$I^2 = \{\text{all polynomials with no constant term and no } x\text{term}\}$$

My Problem

Consider the ring with rational coefficients and 6 variables,arranged into a matrix:

$$\mathbb{Q}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$$

Now consider the ideal $I = (vz - wy, wx - uz, uy - vx)$, that is, the **ideal generated by** 2×2 **minors of the matrix** $\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$

By Hochster and Eagon, the quotient ring

$$\mathbb{Q}\begin{bmatrix} w & u & v \\ x & y & z \end{bmatrix} / (vz - wy, wx - uz, uy - vx)$$

is **Cohen Macaulay!**
However, the rings R/I^t, where $t > 1$, are known to *not* Cohen Macaulay. My project centers around understanding the other local cohomology modules.

References

M. Hochster and J. Eagon, *Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci*, Amer. J. Math. **93** (1971), 1020–1058.