Commutative Algebra and Local Cohomology

Jenny Kenkel

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Looking for Structure

- the integers $\mathbb{Z}$

\[\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\]
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- the rational numbers, $\mathbb{Q}$

example: $1, \frac{3}{4}, \frac{17}{266}, \ldots$
Looking for Structure

▶ the integers \( \mathbb{Z} \)

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▶ the rational numbers, \( \mathbb{Q} \)

example: \( 1, \frac{3}{4}, \frac{17}{266}, \ldots \)

▶ rational polynomials in \( x \) (rational numbers with whole powers of \( x \))

example.: \( 1, 7 + 2x, \frac{3}{4}x^2 - 17x^{10} - \frac{2}{3}x^{11} \)

non example: \( \frac{1}{x}, \sqrt{x}, y \)

We call these \( \mathbb{Q}[x] \)
A ring is a collection of objects that you can add, subtract, and multiply, and always stay in that collection of numbers. However, you can’t necessarily divide.

- The integers, \( \mathbb{Z} \):
  \(-3, -2, -1, 0, 1, 2, 3, \ldots \)

- The rational numbers, \( \mathbb{Q} \):
  \(1, \frac{3}{4}, \frac{17}{266}, \ldots \)

- Rational polynomials, \( \mathbb{Q}[x] \):
  \(1, 7 + 2x, \frac{3}{4}x^2 - 17x^{10} - \frac{2}{3}x^{11}\)

- Not a ring: The set of numbers 0, 1 and 2
  \((1 + 2 = 3, \text{ not in the collection!})\)
Integers, \( \mathbb{Z} \), AND the square root of \(-5\) are all numbers that look like:

\[(\text{integer}) + (\text{integer})\sqrt{-5}:\]

For example,

\[1 + \sqrt{-5} \quad 2\sqrt{-5} \quad 3 - 6\sqrt{-5}\]

We can still add, subtract, and multiply

\[(1 + \sqrt{-5})(2\sqrt{-5}) = \]
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This ring is referred to as \( \mathbb{Z}[\sqrt{-5}] \)
In the integers ($\mathbb{Z}$) a number can be factored into a set of prime (only divisible by itself and 1) numbers.

$$12 = 2 \times 2 \times 3$$

We can rearrange, but we can’t choose different primes. In $\mathbb{Z}[\sqrt{-5}]$, what *is* and *isn’t* a prime is not so clear...
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$$1 + \sqrt{-5} ???$$
Non Unique Factorization

It turns out

\[ 1 + \sqrt{-5} \]

is prime (only divisible by itself and 1) in \( \mathbb{Z}[\sqrt{-5}] \). So are 3 and 2.

\[ 6 = 2 \times 3 \]

but also ...

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$$1 + 5 = 6$$

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \times 3$$
Unique Factorization Domains

- Rings like \( \mathbb{Z} \), where numbers can be uniquely factored, are called **Unique Factorization Domains**.

- Rings like \( \mathbb{Z}[\sqrt{-5}] \) are not **Unique Factorization Domains**.

- The study of **local cohomology** was invented to answer a question about Unique Factorization Domains.

- Exactly how local cohomology relates to UFD’s is outside the scope of this talk.
Local Cohomology

The local cohomologies of a ring are a sequence that starts counting at 0 and tells you something about the ring.

- The $0^{th}$ local cohomology of $\mathbb{Q}$ is $\mathbb{Q}$
- The $1^{st}$ local cohomology of $\mathbb{Q}$ is 0
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Size of Rings

The last nonzero local cohomology of a ring measures how big the ring is.

- $\mathbb{Q}$ (e.g. $1$, $\frac{3}{4}$, $\frac{17}{266}$)
- $\mathbb{Q}[x]$ (e.g. $1$, $7 + 2x$, $\frac{3}{4}x^2 - 17x^{10} - \frac{2}{3}x^{11}$)
- $\mathbb{Q}[x, y]$ (e.g. $1$, $7 + 2x$, $2x^2 + y^3 + 6xy$)

Everything in $\mathbb{Q}$ is also in $\mathbb{Q}[x]$, and everything in $\mathbb{Q}[x]$ is also in $\mathbb{Q}[x, y]$. 

Diagram:

- $\mathbb{Q}$
- $\mathbb{Q}[x]$ in $\mathbb{Q}$
- $\mathbb{Q}[x, y]$ in $\mathbb{Q}[x]$
The last nonzero local cohomology of $\mathbb{Q}$ is the 0th one

The last nonzero local cohomology of $\mathbb{Q}[x]$ is the 1st one

The last nonzero local cohomology of $\mathbb{Q}[x, y]$ is the 2nd one
Consider the ring of polynomials where we can use $x^4, x^3 y, xy^3$, and $y^4$ as variables, and rational numbers as coefficients, e.g.

$$\frac{1}{2}x^4 + x^3 y + xy^3, \quad x^7 y, \quad x^4 y^4$$

Everything in this ring is a polynomial with $x$ and $y$ (that is, $\mathbb{Q}[x, y]$); but this weird ring doesn’t have $x$, $y$, $x^2 y^2$, etc. in it.
Local Cohomology Measures Weirdness

How far the first nonzero local cohomology is from the last nonzero local cohomology measures the weirdness of the ring.

\[ \mathbb{Q}[x, y] \text{ has one nonzero local cohomology} \]

\[ \begin{align*}
\mathbb{Q}[x^4, x^3y, xy^3, y^4] & \text{ has two nonzero local cohomologies} \\
\end{align*} \]
Thank You!