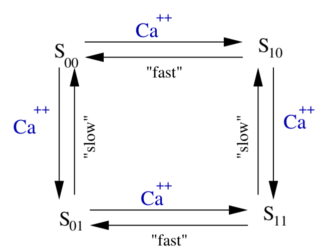
Exercises for Module 1

1. A monomer (represented by A_1) polymerizes to form polymer of length n, (denoted A_n) via the reaction scheme
$$A_n + A_1 \rightleftarrows A_{n+1} \qquad (1)$$
   (a) Use the law of mass action to write a system of differential equations for the dynamics of A_n. What is the equation governing the dynamics of A_1? Check to be sure that $\sum_n n\frac{dA_n}{dt} = 0$.
   (b) Assuming that $\sum_n nA_n = A_0$, find the steady state distribution of polymer lengths.

2. Use the law of mass action to find differential equations governing S_{jk} for the reaction scheme



Use that $S_{00} + S_{01} + S_{10} + S_{11} = 1$.

   (a) Suppose the "top" and "bottom" reactions are independent from the "left" and "right" reactions. Assume that $S_{10} = mh$, $S_{00} = (1-m)h$, $S_{11} = m(1-h)$, $S_{01} = (1-m)(1-h)$. Find the differential equations governing the dynamics of m and h.
   (b) Assume that the top and bottom reactions are fast compared to the left and right reactions. Use the quasi-steady state assumption to find S_{01} in terms of $h = S_{00} + S_{01}$ and find the equation governing the dynamics of h. (This problem is the most difficult in this set. For help, look at Keener and Sneyd, Mathematical Physiology.)

3. Sketch the phase portrait for the system of differential equations
$$\frac{d\phi}{dt} = A\phi(1-\phi)(\phi - a) - h + I_0, \qquad \frac{dh}{dt} = \epsilon(\phi - \gamma h) \qquad (2)$$
where all parameters are positive, $0 < a < \frac{1}{2}$ and $\epsilon << 1$. What qualitatively different kinds of phase portraits are possible (there are 3)?

   Write a simple Matlab code to simulate the solution of this equation (Use $A = 10$, $a = 0.1$, $\epsilon = 0.1$ for starters). Find parameter values for each of the qualitatively different behaviors.

1 Solutions

1. (a) The differential equations are

$$\frac{dA_n}{dt} = k_+A_{n-1}A_1 - k_+A_nA_1 + k_-A_{n+1} - k_-A_n \tag{3}$$

for $n \geq 2$ and

$$\frac{dA_1}{dt} = -2k_+A_1^2 + 2k_-A_2 + \sum_{n=3}^{\infty} k_-A_n - \sum_{n=2}^{\infty} k_+A_nA_1 \tag{4}$$

To check this, note that

$$\begin{aligned}
\sum_{n=2}^{\infty} n\frac{dA_n}{dt} &= \sum_{n=2}^{\infty} n(k_+A_{n-1}A_1 - k_+A_nA_1 + k_-A_{n+1} - k_-A_n) \\
&= \sum_{n=2}^{\infty} nk_+A_{n-1}A_1 - \sum_{n=2}^{\infty} nk_+A_nA_1 + \sum_{n=2}^{\infty} nk_-A_{n+1} - \sum_{n=2}^{\infty} nk_-A_n \\
&= \sum_{n=1}^{\infty}(n+1)k_+A_nA_1 - \sum_{n=2}^{\infty} nk_+A_nA_1 + \sum_{n=3}^{\infty}(n-1)k_-A_n - \sum_{n=2}^{\infty} nk_-A_n \\
&= 2k_+A_1^2 + \sum_{n=2}^{\infty} k_+A_nA_1 - \sum_{n=3}^{\infty} k_-A_n - 2k_-A_2
\end{aligned} \tag{5}$$

so that $\sum_{n=1}^{\infty} n\frac{dA_n}{dt} = 0$.

(b) To find the steady state solution, notice that the equation (3) in steady state ($\frac{dA_n}{dt} = 0$) is a linear difference equation (with A_1 fixed). Therefore, for $n \geq 2$, $A_n = \alpha\mu^n$, where

$$k_+A_1 - k_-\mu - \mu(k_+A_1 - k_-\mu) = 0 \tag{6}$$

so that $\mu = \frac{k_+A_1}{k_-}$. Notice that for consistency, $A_1 = \alpha\mu$ so that $\alpha = \frac{k_-}{k_+}$.
Now to find A_1 we only need to solve the equation $\sum_{n=2}^{\infty} A_n + A_1 = A_0$. However,

$$\sum_{n=2}^{\infty} nA_n = \alpha\sum_{n=2}^{\infty} n\mu^n \tag{7}$$

$$= \alpha\sum_{n=2}^{\infty} n\mu^n = \alpha\left(\frac{1}{(1-\mu)^2} - \mu\right) \tag{8}$$

This leaves us with a single equation for μ

$$\frac{k_-\mu}{k_+}\frac{1}{(1-\mu)^2} = A_0. \tag{9}$$