Math 6780 Homework Exercises, Spring 2012

1 Homework Set 1, Due Feb. 23, 2012.

1. Consider the following three state model for sodium channels:

$$I \xleftarrow{\delta} C \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} O \xrightarrow{\gamma} I \tag{1}$$

   (a) Calculate the splitting probability (i.e., the probability that the state I is entered from state O or state C), the expected time to enter state I, and the expected number of times state O is entered before going into state I, assuming in all cases that the process is initially in state C.

   (b) Simulate this process using the Gillespie algorithm using parameters $\alpha = 1/\text{ms}$, $\beta = 0.4/\text{ms}$, $\gamma = 1.6/\text{ms}$ and $\delta = 1/\text{ms}$. Collect data from a large number of simulations and verify the analytical calculations done for the first part of this problem.

2. Consider a birth-death process in which the death rate β is constant ($k_n^- = n\beta$) and the birth rate is population size dependent, decreasing as a function of population size, $k_n^+ = \alpha(N - n)$.

   (a) Write the master equation for this process. Find the steady state solution (a binomial distribution).

   (b) Find the equation for the generating function for this process.

   (c) Find the steady solution of the generating function equation.

   (d) Simulate this process using the Gillespie algorithm. Compare the results of the simulation with the steady distribution you found using the master equation.

2 Homework Set 2, Due March 22, 2012

1. Find the mean first exit time for a particle starting at position $x = 0$ from the piecewise linear potential

$$U(x) = \begin{cases} -\frac{\Delta G x}{L} & -L < x < 0 \\ \frac{\Delta G x}{L} & 0 < x < L \end{cases}, \tag{2}$$

   with a reflecting boundary at $x = -L$ and absorbing boundary at $x = L$, and with a force F. Assume the particle has drag coefficient ξ.

2. Suppose a particle can diffuse freely along a line of length L, but there are binding sites to which it can bind at two positions $x = a$ and $x = b$, $0 < a < b < L$. Find the splitting probability for binding, i.e., the probability that binding will occur at site a and the probability that binding will occur at site b for a particle that starts at $x = 0$. Assume that the binding affinities for the two binding sites are the same (and modeled as delta functions) and that the particle cannot escape through $x = 0$ but it can escape through $x = L$.

3. Suppose that there are three states underlying the random variable X (with values $x = 0, 1, 2$) with transitions

$$S_0 \underset{\beta}{\overset{2\alpha}{\rightleftarrows}} S_1 \underset{2\beta}{\overset{\alpha}{\rightleftarrows}} S_2. \tag{3}$$

(a) Use the Gillespie algorithm to find sample paths for the solution of

$$\frac{dy}{dt} = x \tag{4}$$

Use this to estimate the mean and the variance as a function of time for this process.

(b) What is the Master equation for this stochastic differential equation?

(c) Suppose that α and β are much greater than 1. Use the rapid equilibrium approximation to find the Fokker Planck equation and the equivalent Langevin equation for this process. Compare the result of this calculation with the numerical estimates found in part a.