

Notes on Adiabatic Reduction of Master Equations

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1 Background on Stochastic Models

In this section we examine the stochastic differential equation

$$\frac{dy}{dt} = f(y, x) \quad (1)$$

where x is a discrete random variable taking on values x_k , $k = 0, 1, \dots, n$. The process for x is a Markov process, and there are transition probabilities $W_{j,k}$ which is the transition probability of going from state k to state j . Following standard arguments, the Fokker-Planck equation for the probability distribution function $p(y, x, t)$ is (following Gardiner)

$$\frac{\partial p_k}{\partial t} = -\frac{\partial}{\partial y}(f_k p_k) + \sum_j (W_{j,k} p_j - W_{j,k} p_k) \quad (2)$$

where $p_k = p(y, x_k, t)$ and $f_k = f(y, x_k)$. We can write this in vector notation as

$$\frac{\partial \mathbf{p}}{\partial t} = -\frac{\partial}{\partial y}(\mathbf{F}\mathbf{p}) + \mathbf{A}\mathbf{p}. \quad (3)$$

where \mathbf{F} is the diagonal matrix with elements f_k .

It also follows that the backward Fokker-Planck equation is

$$\frac{\partial q_k}{\partial t} = -f_k \frac{\partial q_k}{\partial y} - \sum_j W_{j,k} (q_k - q_j) \quad (4)$$

which in vector form is

$$\frac{\partial \mathbf{q}}{\partial t} = -\mathbf{F} \frac{\partial \mathbf{q}}{\partial y} - \mathbf{A}^T \mathbf{q}. \quad (5)$$

To find the mean first exit time from some region, say $a < y < b$, we let

$$G_j(z, t) = \int_a^b \sum_k p_k(y, t | z, x_j, 0) dy = P(T \geq t), \quad (6)$$

where T is the exit time, starting from initial position z , x_j (so we denote it as $T_j(z)$). Note that $G_j(z, 0) = 1$. Because this process is autonomous,

$$p_k(y, t|z, x_j, 0) = p_k(y, 0|z, x_j, -t) = q_j(z|y, k), \quad (7)$$

which is governed by the backward F-P equation

$$\frac{\partial q_j}{\partial t} = f_j \frac{\partial q_j}{\partial z} + \sum_i W_{i,j}(q_j - q_i). \quad (8)$$

so that

$$\frac{\partial G_j}{\partial t} = f_j \frac{\partial G_j}{\partial z} + \sum_i W_{i,j}(G_j - G_i). \quad (9)$$

Now observe that

$$P(T_j(z) \leq t) = 1 - G_j(z, t), \quad (10)$$

so that

$$\langle T_j(z) \rangle = - \int_0^\infty t \frac{\partial G_j}{\partial t} dt = \int_0^\infty G_j(z, t) dt. \quad (11)$$

It follows from (9) that

$$-1 = f_j \frac{\partial \langle T_j(z) \rangle}{\partial z} + \sum_i W_{i,j}(\langle T_j(z) \rangle - \langle T_i(z) \rangle). \quad (12)$$

with appropriate boundary conditions.

2 Adiabatic Reduction

We can reduce the F-P system to a standard F-P equation in the limit that the transitions between x_k states are fast. If this is the case, then we can write the F-P system (5) as

$$\frac{\partial \mathbf{p}}{\partial t} = - \frac{\partial}{\partial y} (\mathbf{F} \mathbf{p}) + \frac{1}{\epsilon} A \mathbf{p}. \quad (13)$$

Now, it must be that the matrix A has a zero eigenvalue with eigenvector ϕ . The corresponding left eigenvector is ψ with entries $(\psi_j) = 1$. We assume that $\langle \phi, \psi \rangle = 1$. Using these, we split \mathbf{p} into two parts

$$\mathbf{p} = v\phi + w \quad (14)$$

where $\langle \mathbf{p}, \psi \rangle = v$, and $\langle w, \psi \rangle = 0$. It follows that

$$\frac{\partial v}{\partial t} = -\psi^T \frac{\partial}{\partial y} (\mathbf{F}(v\phi + w)), \quad (15)$$

and

$$\frac{\partial w}{\partial t} = \frac{1}{\epsilon} A w - \frac{\partial}{\partial y} (\mathbf{F} \mathbf{p}) + \psi^T \frac{\partial}{\partial y} (\mathbf{F} \mathbf{p}) \phi. \quad (16)$$

Here, the fast behavior of w is evident, so we take w to be in quasi-steady state. Thus, we take

$$Aw = \epsilon \frac{\partial}{\partial y}(v\mathbf{F}\phi) - \epsilon\psi^T \frac{\partial}{\partial y}(v\mathbf{F}\phi)\phi + O(\epsilon^2). \quad (17)$$

This equation can be solved uniquely for w subject to the constraint $\psi^T w = 0$; we denote this as

$$w = \epsilon A^\perp \frac{\partial}{\partial y}(v\mathbf{F}\phi) - \epsilon A^\perp \psi^T \frac{\partial}{\partial y}(v\mathbf{F}\phi)\phi + O(\epsilon^2), \quad (18)$$

where A^\perp is the inverse of the properly constrained A . Consequently,

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}(\psi^T \mathbf{F}v\phi) - \frac{\partial}{\partial y} \left(\epsilon \psi^T \mathbf{F}A^\perp \left(\frac{\partial}{\partial y}(v\mathbf{F}\phi) - \psi^T \frac{\partial}{\partial y}(v\mathbf{F}\phi)\phi \right) \right), \quad (19)$$

2.1 Examples

2.1.1

Consider the simple example

$$\frac{dy}{dt} = kx \quad (20)$$

where x is either 0 or 1, with transition probabilities α and β . The F-P equation is

$$\frac{\partial p}{\partial t} = \alpha q - \beta p - \frac{\partial kp}{\partial y} \quad (21)$$

$$\frac{\partial q}{\partial t} = \beta p - \alpha q \quad (22)$$

We can reduce this to a single pde by observing that $\frac{\partial q}{\partial t} = -\frac{\partial p}{\partial t} - \frac{\partial ap}{\partial y}$ so that

$$p_{tt} = -(\alpha + \beta)p_t - \alpha kp_y - kp_{ty}. \quad (23)$$

I do not know how to solve this rather curious parabolic equation. However, we can find the first few moments quite easily. Define $M_k = \int y^k p dy$. Then,

$$\frac{d^2 M_0}{dt^2} + (\alpha + \beta) \frac{dM_0}{dt} = 0. \quad (24)$$

$$\frac{d^2 M_1}{dt^2} + (\alpha + \beta) \frac{dM_1}{dt} = \alpha k M_0 + k \frac{dM_0}{dt}. \quad (25)$$

and

$$\frac{d^2 M_2}{dt^2} + (\alpha + \beta) \frac{dM_2}{dt} = 2\alpha k M_1 + 2k \frac{dM_1}{dt}. \quad (26)$$

Clearly, $M_0 \rightarrow \text{constant}$, so we take M_0 to be a constant. Next we observe that M_1 approaches

$$M_1 = \frac{k\alpha}{\alpha + \beta} M_0 \quad (27)$$

and M_2 approaches (recheck this!)

$$M_2 = 2k \frac{\alpha\beta}{(\alpha + \beta)^3} M_0 t + k \frac{\alpha^2}{(\alpha + \beta)^2} M_0 t^2 \quad (28)$$

Suppose that α and β are large compared to k . Then the exchange between states is fast relative to the rate of change of y , and we should be able to do a qss reduction. To do so, we introduce dimensionless time (set $k = 1$) and let $\epsilon = \frac{1}{\alpha + \beta}$, and introduce $a = \frac{\alpha}{\alpha + \beta}$ and $b = \frac{\beta}{\alpha + \beta}$, so that $a + b = 1$. In terms of these variables, the F-P equations are

$$\frac{\partial p}{\partial t} = \frac{1}{\epsilon}(aq - bp) - \frac{\partial p}{\partial y}, \quad (29)$$

$$\frac{\partial q}{\partial t} = \frac{1}{\epsilon}(bp - aq). \quad (30)$$

We now introduce the change of variables

$$v = p + q, \quad w = bp - aq, \quad (31)$$

so that

$$p = av + w, \quad q = bv - w. \quad (32)$$

In terms of these variables the F-P equations are

$$\frac{\partial v}{\partial t} = -\frac{\partial av}{\partial y} - \frac{\partial w}{\partial y} \quad (33)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\epsilon}w - b\frac{\partial av}{\partial y} - b\frac{\partial w}{\partial y} \quad (34)$$

Now we see the obvious fast-slow separation and take w to be in quasi-steady state, so that

$$w = -\epsilon b \frac{\partial av}{\partial y} + O(\epsilon^2), \quad (35)$$

from which it follows that

$$\frac{\partial v}{\partial t} = -\frac{\partial av}{\partial y} + \frac{\partial}{\partial y}(\epsilon b \frac{\partial av}{\partial y}) + O(\epsilon^2), \quad (36)$$

which is the standard F-P equation we seek.

For the more general problem

$$\frac{dy}{dt} = xf(y) - g(y), \quad (37)$$

the F-P equations are

$$\frac{\partial p}{\partial t} = \frac{1}{\epsilon}(aq - bp) - \frac{\partial}{\partial y}((f - g)p), \quad (38)$$

$$\frac{\partial q}{\partial t} = \frac{1}{\epsilon}(bp - aq) + \frac{\partial}{\partial y}(gq). \quad (39)$$

We now introduce the change of variables (31) and find

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}((af - g)v + fw), \quad (40)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\epsilon}w - a\frac{\partial}{\partial y}(gq) - b\frac{\partial}{\partial y}((f - g)p). \quad (41)$$

Again, the fast-slow separation is apparent and we take w to be in quasi-steady state, so that

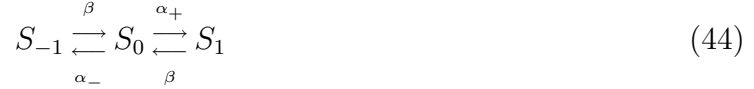
$$w = \epsilon(-a\frac{\partial}{\partial y}(gbv) - b\frac{\partial}{\partial y}((f - g)av)) + O(\epsilon^2), \quad (42)$$

from which it follows that

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}((af - g)v) + \frac{\partial}{\partial y} \left(\epsilon fb \frac{\partial}{\partial y}(fav) + \epsilon fgv \frac{\partial b}{\partial y} \right) + O(\epsilon^2), \quad (43)$$

which is the F-P equation we seek.

A second interesting example is the three state process with $x = -1, 0, 1$ and transitions



There are numerous physical situations where this might occur.

The master equations are

$$\frac{\partial p_{-1}}{\partial t} = \alpha_- p_0 - \beta p_{-1} + V \frac{\partial p_{-1}}{\partial x}, \quad (45)$$

$$\frac{\partial p_0}{\partial t} = \beta p_{-1} + \beta p_1 - (\alpha_- + \alpha_+) p_0 + D \frac{\partial^2 p_0}{\partial x^2}, \quad (46)$$

$$\frac{\partial p_1}{\partial t} = \alpha_+ p_0 - \beta p_1 - V \frac{\partial p_1}{\partial x}. \quad (47)$$

We assume that α_- , α_+ , and β are large so introduce the scaled parameters $a_{\pm} = \epsilon\alpha_{\pm}$, $b = \epsilon\beta$ where $\epsilon = \frac{1}{\alpha_- \alpha_+ + \beta}$, som that $a_- + a_+ + b = 1$. Then, the equations become

$$\frac{\partial p_{-1}}{\partial t} = \frac{1}{\epsilon}(a_- p_0 - b p_{-1}) + V \frac{\partial p_{-1}}{\partial x}, \quad (48)$$

$$\frac{\partial p_0}{\partial t} = \frac{1}{\epsilon}(b p_{-1} + b p_1 - (a_- + a_+) p_0) + D \frac{\partial^2 p_0}{\partial x^2}, \quad (49)$$

$$\frac{\partial p_1}{\partial t} = \frac{1}{\epsilon}(a_+ p_0 - b p_1) - V \frac{\partial p_1}{\partial x}. \quad (50)$$

The nullspace of the matrix is spanned by the vector

$$\phi = \begin{pmatrix} a_- \\ b \\ a_+ \end{pmatrix}. \quad (51)$$

We set

$$p = \begin{pmatrix} p_{-1} \\ p_0 \\ p_1 \end{pmatrix} = v\phi + \begin{pmatrix} w_{-1} \\ w_0 \\ w_1 \end{pmatrix}, \quad (52)$$

and find the equation for v by projecting

$$\frac{\partial v}{\partial t} = V \frac{\partial p_{-1}}{\partial x} - V \frac{\partial p_1}{\partial x} + D \frac{\partial^2 p_0}{\partial x^2} \quad (53)$$

$$= V(a_- - a_+) \frac{\partial v}{\partial x} + V \frac{\partial}{\partial x} (w_{-1} - w_1) + D \frac{\partial^2}{\partial x^2} (bv + w_0) \quad (54)$$

The vector w must satisfy the equation

$$\frac{1}{\epsilon} \begin{pmatrix} -b & a_- & 0 \\ b & -(a_+ a_-) & b \\ 0 & a_+ & b \end{pmatrix} w = \begin{pmatrix} -V a_- \frac{\partial v}{\partial x} \\ D b \frac{\partial^2 v}{\partial x^2} \\ V a_+ \frac{\partial v}{\partial x} \end{pmatrix} - C \begin{pmatrix} a_- \\ b \\ a_+ \end{pmatrix}, \quad (55)$$

where $C = -V(a_- - a_+) \frac{\partial v}{\partial x} + D b \frac{\partial^2 v}{\partial x^2}$. The solution of this problem is given by (ignoring D)

$$w_{-1} = \frac{\epsilon}{b} V a_- v_x (b^2 + 2a_+ b + 2a_+), \quad (56)$$

$$w_0 = -\epsilon b V v_x (-a_+ + a_-), \quad (57)$$

$$w_1 = -\frac{\epsilon}{b} V a_+ v_x (b^2 + 2a_- b + 2a_-), \quad (58)$$

so that

$$w_{-1} - w_1 = \frac{\epsilon}{b} V ((a_- + a_+) b^2 + 4b a_+ a_- + 2a_+ a_-) v_x \quad (59)$$

Thus, the effective Fokker-Planck equation (ignoring all higher order derivatives) is

$$\frac{\partial v}{\partial t} = V(a_- - a_+) \frac{\partial v}{\partial x} + D_{eff} \frac{\partial^2 v}{\partial x^2} \quad (60)$$

where

$$D_{eff} = bD + \frac{\epsilon}{b} V^2 ((a_- + a_+) b^2 + 4b a_+ a_- + 2a_+ a_-) \quad (61)$$

2.2 Higher Dimensional example

Consider the movement of a particle on a random microtubule network in 2-D. The random variables are the position and the angle of movement θ , so we seek the pdf $p(x, y, \theta, t)$. The Master Equations are

$$\frac{\partial p_b}{\partial t} = -\beta p_b + \alpha q(\theta, x) p_u - V \frac{\partial}{\partial x} (\cos \theta p_b) - V \frac{\partial}{\partial y} (\sin \theta p_b) \quad (62)$$

$$\frac{\partial p_u}{\partial t} = \beta p_b - \alpha q(\theta, x) p_u + D \frac{\partial^2 p_u}{\partial x^2} + D \frac{\partial^2 p_u}{\partial y^2} \quad (63)$$

Rescaling, with $b = \frac{\beta}{\beta+\alpha q}$ and $a = \frac{\alpha}{\beta+\alpha q}$ (so that $a + b = 1$), we have

$$\frac{\partial p_b}{\partial t} = \frac{1}{\epsilon}(-bp_b + ap_u) - V \frac{\partial}{\partial x}(\cos \theta p_b) - V \frac{\partial}{\partial y}(\sin \theta p_b) \quad (64)$$

$$\frac{\partial p_u}{\partial t} = \frac{1}{\epsilon}(bp_b - ap_u) + D \frac{\partial^2 p_u}{\partial x^2} + D \frac{\partial^2 p_u}{\partial y^2}. \quad (65)$$

The adiabatic analysis of this problem is actually easier than for the 1-D problem. We set

$$\begin{pmatrix} p_b \\ p_u \end{pmatrix} = v \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} w \\ -w \end{pmatrix} \quad (66)$$

Projecting, the equation for v is

$$\frac{\partial v}{\partial t} = -V \frac{\partial}{\partial x}(\cos \theta p_b) - V \frac{\partial}{\partial y}(\sin \theta p_b) + D \frac{\partial^2 p_u}{\partial x^2} + D \frac{\partial^2 p_u}{\partial y^2} \quad (67)$$

and the equation for w is

$$\frac{\partial w}{\partial t} = -\frac{1}{\epsilon}w + b \left(-V \frac{\partial}{\partial x}(\cos \theta p_b) - V \frac{\partial}{\partial y}(\sin \theta p_b) \right) - a \left(D \frac{\partial^2 p_u}{\partial x^2} + D \frac{\partial^2 p_u}{\partial y^2} \right) \quad (68)$$

It follows that the qss solution for w is (ignoring D)

$$w = -V\epsilon b \left(\frac{\partial}{\partial x}(\cos \theta av) + \frac{\partial}{\partial y}(\sin \theta av) \right), \quad (69)$$

leading to the Fokker Planck equation

$$\frac{\partial v}{\partial t} = -V \frac{\partial}{\partial x}(\cos \theta av) - V \frac{\partial}{\partial y}(\sin \theta av) \quad (70)$$

$$V^2 \frac{\partial}{\partial x}(\cos \theta \epsilon b \frac{\partial}{\partial x}(\cos \theta av)) + V^2 \frac{\partial}{\partial x}(\cos \theta \epsilon b \frac{\partial}{\partial y}(\sin \theta av)) \quad (71)$$

$$V^2 \frac{\partial}{\partial y}(\sin \theta \epsilon b \frac{\partial}{\partial y}(\cos \theta av)) + V^2 \frac{\partial}{\partial y}(\sin \theta \epsilon b \frac{\partial}{\partial x}(\sin \theta av)) + D \frac{\partial^2 av}{\partial x^2} + D \frac{\partial^2 av}{\partial y^2} \quad (72)$$

which seems to be a reasonable extension of the 1-D example above.

2.2.1

Now let's apply this technology to a more general problem. Let's consider the stochastic differential equation

$$\frac{dy}{dt} = x_k f(y) - g(y), \quad (73)$$

where x_k can be an integer between 0 and N . We suppose that x_k represents the number of open channels and that channels act independently. (We can also consider cooperativity, but this may come later). Thus, the transition probabilities are

$$P(x_k \rightarrow x_{k+1}) = (N - k)\alpha, \quad P(x_k \rightarrow x_{k-1}) = k\beta. \quad (74)$$

Thus,

$$W_{k+1,k} = (N - k)\alpha, \quad W_{k-1,k} = k\beta, \quad (75)$$

and all other transitions are impossible. The F-P equations are then

$$\frac{\partial p}{\partial t} = \frac{1}{\epsilon}Ap - \frac{\partial}{\partial y}(Bp), \quad (76)$$

where

$$a_{k,k-1} = a(N - k + 1), \quad a_{k,k} = -a(N - k) - bk, \quad a_{k,k+1} = b(k + 1), \quad (77)$$

for $k = 0, \dots, N$, and all other entries of A are zero, and where B is the diagonal matrix with entries $b_{k,k} = kf(y) - g(y)$.

Now to do the qss analysis, we observe that A has a one-dimensional nullspace spanned by the vector ϕ , and a one-dimensional nullspace spanned by ψ where

$$\phi_k = \binom{N}{k} a^k b^{N-k}, \quad \psi_k = 1. \quad (78)$$

We set $p = v\phi + w$, where $v = \langle p, \psi \rangle$ and $\langle w, \psi \rangle = 0$. Then,

$$\frac{\partial v}{\partial t} = -\psi^T \frac{\partial}{\partial y}(B(v\phi + w)), \quad (79)$$

and

$$\frac{\partial w}{\partial t} = \frac{1}{\epsilon}Aw - \frac{\partial}{\partial y}(Bp) + \psi^T \frac{\partial}{\partial y}(Bp)\phi, \quad (80)$$

Here the qss approximation is apparent. We take

$$Aw = \epsilon \frac{\partial}{\partial y}(Bv\phi) - \epsilon \psi^T \frac{\partial}{\partial y}(Bv\phi)\phi + O(\epsilon^2), \quad (81)$$

with $\psi^T w = 0$. Observe, that w is uniquely determined by this equation, although A is not invertible. Now the interesting question is if we can find analytical expressions for w .

Using Maple, I have been able to determine that

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}((Naf - g)v) + N \frac{\partial}{\partial y} \left(\epsilon b f \frac{\partial}{\partial y}(afv) + \epsilon(Ng - (N-1)af)fv \frac{\partial b}{\partial y} \right). \quad (82)$$

When $f = \frac{1}{N}$, $g = 0$, this reduces to

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y} \left(\left(1 + \epsilon \left(1 - \frac{1}{N} \right) \frac{\partial b}{\partial y} \right) av \right) + \frac{\partial}{\partial y} \left(\frac{\epsilon b}{N} \frac{\partial}{\partial y}(av) \right). \quad (83)$$

2.2.2 More Complicated Markov Processes

Suppose we have a more complicated gating behavior, such as

$$\frac{dy}{dt} = x_1 x_2 f(y) - g(y) \quad (84)$$

where both x_1 and x_2 are independent, identical, discrete random variables taking on values 0 and 1. Now there are four functions $p_{j,k}$ to be determined, and the Fokker-Planck equations are

$$\frac{\partial p_{00}}{\partial t} = -2\alpha p_{00} + \beta p_{10} + \beta p_{01} + \frac{\partial}{\partial y}(gp_{00}) \quad (85)$$

$$\frac{\partial p_{10}}{\partial t} = \alpha p_{00} - (\beta + \alpha)p_{10} + \beta p_{11} + \frac{\partial}{\partial y}(gp_{10}) \quad (86)$$

$$\frac{\partial p_{01}}{\partial t} = \alpha p_{00} - (\beta + \alpha)p_{01} + \beta p_{11} + \frac{\partial}{\partial y}(gp_{01}) \quad (87)$$

$$\frac{\partial p_{11}}{\partial t} = \alpha p_{10} + \alpha p_{01} - 2\beta p_{11} - \frac{\partial}{\partial y}((f - g)p_{11}) \quad (88)$$

We want to examine this in the limit that α and β are large, and so set (as before) $a = \frac{\alpha}{\alpha+\beta}$, $b = \frac{\beta}{\alpha+\beta}$, and $\epsilon = \frac{1}{\alpha+\beta}$, with the result that

$$\frac{\partial p_{00}}{\partial t} = \frac{1}{\epsilon}(-2ap_{00} + bp_{10} + bp_{01}) + \frac{\partial}{\partial y}(gp_{00}) \quad (89)$$

$$\frac{\partial p_{10}}{\partial t} = \frac{1}{\epsilon}(ap_{00} - p_{10} + bp_{11}) + \frac{\partial}{\partial y}(gp_{10}) \quad (90)$$

$$\frac{\partial p_{01}}{\partial t} = \frac{1}{\epsilon}(ap_{00} - p_{01} + bp_{11}) + \frac{\partial}{\partial y}(gp_{01}) \quad (91)$$

$$\frac{\partial p_{11}}{\partial t} = \frac{1}{\epsilon}(ap_{10} + ap_{01} - 2bp_{11}) - \frac{\partial}{\partial y}((f - g)p_{11}) \quad (92)$$

Now we want to apply the technology developed above to this system of equations. To do so we examine the matrix

$$A = \begin{pmatrix} -2a & b & b & 0 \\ a & -1 & 0 & b \\ a & 0 & -1 & b \\ 0 & a & a & -2b \end{pmatrix} \quad (93)$$

The nullspace of A is spanned by

$$\phi = \begin{pmatrix} b^2 \\ ab \\ ab \\ a^2 \end{pmatrix} \quad (94)$$

so we introduce the change of variables

$$v = \langle \psi, p \rangle, \quad w = p - \langle \psi, p \rangle \phi, \quad (95)$$

so that

$$p = v\phi + w \quad (96)$$

and find

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}((a^2 f - g)v) + \frac{\partial}{\partial y} \left(\frac{\epsilon}{2} f \left(b(3a + 1) \frac{\partial}{\partial y} (a^2 f v) - 2gv \frac{\partial a^2}{\partial y} \right) \right). \quad (97)$$

More generally, when there are N channels, we find (using Maple)

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y} (N(a^2 f - g)v) + \frac{\partial}{\partial y} \left(\frac{N\epsilon}{2} f \left(b(3a + 1) \frac{\partial}{\partial y} (a^2 f v) - 2gv \frac{\partial a^2}{\partial y} + (N - 1) f \frac{\partial a^4}{\partial y} \right) \right). \quad (98)$$

2.3 Remarks on modeling

The above calculation requires amplification. As stated, if there are N channels, there are 2^N configurations. However, if the states are independent, a significant reduction can be made, without error (except for a small initial transient). Since each channel can be in one of three states, we let p_{ijk} be the probability that there are i channels in state 0_1 , j channels in state 0_2 and k channels in state 1 (open). Obviously, $i + j + k = N$, and i , j , and k are nonnegative. Now the possible transitions are

$$p_{i,k-i,N-k} \xrightarrow{(k-i)\alpha} p_{i,k-i-1,N-k+1}, \quad k \neq 0 \quad (99)$$

$$p_{i,k-i,N-k} \xrightarrow{(k-i)\beta} p_{i+1,k-i-1,N-k}, \quad k \neq i \quad (100)$$

$$p_{i,k-i,N-k} \xrightarrow{2i\alpha} p_{i-1,k-i+1,N-k}, \quad i \neq 0 \quad (101)$$

$$p_{i,k-i,N-k} \xrightarrow{2(N-k)\beta} p_{i,k-i+1,N-k-1}, \quad k \neq N \quad (102)$$

Furthermore, in state i, j, k , the deterministic dynamics are

$$\frac{dy}{dt} = kf(y) - g(y) \quad (103)$$

2.4 K-step processes

Suppose there are K closed states,

$$C_k \leftrightarrow C_{k-1} \leftrightarrow \dots \leftrightarrow C_1 \leftrightarrow O \quad (104)$$

It is easy to determine that the mean value is

$$E(X) = \frac{a^k}{(a+b)^k} \quad (105)$$

The variance, on the other hand seems not to have such a nice formula. Here we list the first six:

$$V_1 = \frac{ab}{(a+b)^3}, \quad (106)$$

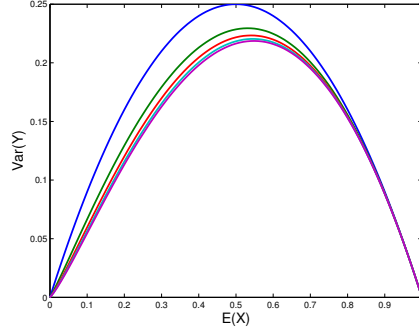


Figure 1: Variance of the process y as a function of K , plotted as a function of the mean of the process X .

$$V_2 = \frac{1}{2} \frac{a^2 b (4a + b)}{(a + b)^5}, \quad (107)$$

$$V_3 = \frac{1}{6} \frac{a^3 b (18a^2 + 9ab + 2b^2)}{(a + b)^7}, \quad (108)$$

$$V_4 = \frac{1}{12} \frac{a^4 b (3b^3 + 16ab^2 + 36a^2b + 48a^3)}{(a + b)^9}, \quad (109)$$

$$V_5 = \frac{1}{60} \frac{a^5 b (200b^2a^2 + 300ba^3 + 300a^4 + 75b^3a + 12b^4)}{(a + b)^{11}}. \quad (110)$$

The important observation is that the variance is a decreasing function of K , and it appears to be converging in the limit of large K .

3 Adiabatic reduction for a Brownian Particle

We begin with the two dimensional equation

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}(up) + \frac{\partial}{\partial u}(U'(y)p) + \gamma \frac{\partial}{\partial u}(up + \frac{\partial p}{\partial u}), \quad (111)$$

where $\gamma = \frac{\beta}{m}$. Set

$$L_1 p = \frac{\partial}{\partial u}(up + \frac{\partial p}{\partial u}), \quad L_2 p = -\frac{\partial}{\partial y}(up) + U'(y) \frac{\partial}{\partial u} p. \quad (112)$$

We want a projection operator based on the null-space of L_1 . Note that $L_1 \phi = 0$ implies that

$$\phi' + u\phi = 0, \quad (113)$$

so that

$$\phi(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}). \quad (114)$$

The adjoint operator has a null space spanned by $\psi = 1$, so the projection operator we seek is

$$Pf = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \int_{-\infty}^{\infty} f(u, y) du. \quad (115)$$

Now we set $p = Pp + (1 - P)p = v + w$ (so that $v = f(y, t)\phi(u)$) and observe that

$$L_2 Pp = -\frac{\partial}{\partial y}(uPp) + U'(y) \frac{\partial Pp}{\partial u} = (f'(y) + f(y)U'(y)) \frac{d\phi}{du} = g(y) \frac{d\phi}{du}, \quad (116)$$

so that $PL_2 Pp = 0$. Furthermore, by construction, $PL_1 = 0$. Thus,

$$\frac{\partial v}{\partial t} = P(\gamma L_1 + L_2)(v + w) = PL_2 w, \quad (117)$$

and

$$\frac{\partial w}{\partial t} = \gamma L_1 w + (1 - P)L_2 v + L_2 w. \quad (118)$$

Now we assume γ is large so take w to be in qss, so take

$$\gamma L_1 w = -(1 - P)L_2 v = -L_2 v, \quad (119)$$

or

$$\gamma \frac{\partial}{\partial u}(uw + \frac{\partial w}{\partial u}) = -g(y) \frac{d\phi}{du}. \quad (120)$$

Thus,

$$\gamma \frac{\partial}{\partial u} \left(\frac{w}{\phi} \right) = -g(y), \quad (121)$$

so that

$$\gamma w = -g(y)u\phi = g(y) \frac{\partial \phi}{\partial u}. \quad (122)$$

Now, notice that

$$L_2 w = \frac{\partial}{\partial y} \left(\frac{g}{\gamma} \right) u^2 \phi - U'(y) \frac{g}{\gamma} \frac{\partial}{\partial u} (u\phi), \quad (123)$$

so that

$$PL_2 w = \phi \frac{\partial}{\partial y} \left(\frac{g}{\gamma} \right) \int_{-\infty}^{\infty} u^2 \phi du = \phi \frac{\partial}{\partial y} \left(\frac{g}{\gamma} \right). \quad (124)$$

It follows that

$$\frac{\partial f}{\partial t} = \frac{1}{\gamma} \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} + U'(y)f \right), \quad (125)$$

the Fokker-Planck equation we were looking for.