Normal Form for the van der Pol equation

The purpose of these notes is to find the normal form for the van der Pol equation

$$\ddot{y} - 2a\dot{y} + y^2\dot{y} + y = 0. \tag{1}$$

First, we write the equation as the system

$$\frac{d}{dt}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} x \\ 2ax - y^2x - y \end{pmatrix} \tag{2}$$

Notice that the linear part of this system is

$$\frac{d}{dt}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = A\begin{pmatrix} y \\ x \end{pmatrix}. \tag{3}$$

The characteristic equation for A is $\lambda^2 - 2a\lambda + 1 = 0$, so the eigenvalues of A are $\lambda = a \pm ib$, where $b = \sqrt{1-a^2}$, and the corresponding eigenvector is $p = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$. We use this to introduce a new complex variable

$$\begin{pmatrix} y \\ x \end{pmatrix} = zp + \overline{zp}. \tag{4}$$

It follows easily that

$$(\overline{\lambda} - \lambda)\dot{z} = (\overline{\lambda} - \lambda)\lambda z + (z + \overline{z})^2(z\lambda + \overline{z}\overline{\lambda}) \tag{5}$$

Now we try to find a transformation that simplifies this as much as possible, by eliminating as many of the cubic terms as possible. We try

$$z = w + \frac{h_{30}}{6}w^3 + \frac{h_{21}}{2}w^2\overline{w} + \frac{h_{12}}{2}w\overline{w}^2 + \frac{h_{03}}{6}\overline{w}^3 \tag{6}$$

with inverse transformation

$$w = z - \frac{h_{30}}{6}z^3 - \frac{h_{21}}{2}z^2\overline{z} - \frac{h_{12}}{2}z\overline{z}^2 - \frac{h_{03}}{6}\overline{z}^3 + O(|z|^4). \tag{7}$$

It follows that

$$\begin{aligned}
\dot{w} &= \dot{z}(1 - \frac{h_{30}}{2}z^2 - h_{21}z\overline{z} - \frac{h_{12}}{2}\overline{z}^2) + \dot{\overline{z}}(-\frac{h_{21}}{2}z^2 - h_{12}z\overline{z} - \frac{h_{03}}{2}\overline{z}^2) + \cdots \\
&= \dot{z} - \dot{z}(\frac{h_{30}}{2}z^2 + h_{21}z\overline{z} + \frac{h_{12}}{2}\overline{z}^2) - \dot{\overline{z}}(\frac{h_{21}}{2}z^2 + h_{12}z\overline{z} + \frac{h_{03}}{2}\overline{z}^2) + \cdots \\
&= (\lambda z + \frac{1}{\overline{\lambda} - \lambda}(z + \overline{z})^2(z\lambda + \overline{z}\overline{\lambda})) - \lambda z(\frac{h_{30}}{2}z^2 + h_{21}z\overline{z} + \frac{h_{12}}{2}\overline{z}^2) - \overline{\lambda}\overline{z}(\frac{h_{21}}{2}z^2 + h_{12}z\overline{z} + \frac{h_{03}}{2}\overline{z}^2) + \cdots \\
&= \lambda w + \frac{1}{\overline{\lambda} - \lambda}(w^3 + (2\lambda + \overline{\lambda})w^2\overline{w} + (\lambda + 2\overline{\lambda})w\overline{w}^2 + \overline{\lambda}\overline{w}^3) \\
&\quad -\lambda\frac{h_{30}}{3}w^3 - (\lambda + \overline{\lambda})\frac{h_{21}}{2}w^2\overline{w} - \overline{\lambda}h_{12}w\overline{w}^2 + (\frac{\lambda}{6} - \frac{\overline{\lambda}}{2})h_{03}\overline{w}^3 + \cdots
\end{aligned}$$

We see that we can eliminate three terms by setting

$$h_{30} = \frac{3}{\lambda(\overline{\lambda} - \lambda)}, h_{12} = \frac{\lambda + 2\overline{\lambda}}{\overline{\lambda}(\overline{\lambda} - \lambda)}, h_{03} = \frac{6\overline{\lambda}}{(3\overline{\lambda} - \lambda)(\overline{\lambda} - \lambda)}, \tag{8}$$

so that

$$\dot{w} = \lambda w + (\frac{2\lambda + \overline{\lambda}}{\overline{\lambda} - \lambda} - (\lambda + \overline{\lambda})\frac{h_{21}}{2})w^2\overline{w}. \tag{9}$$

However, since $\lambda + \overline{\lambda} = 2a$ we cannot use h_{21} to eliminate the fourth term. Instead, we set $h_{21} = \frac{3i}{4b}$ so that

$$\frac{2\lambda + \overline{\lambda}}{\overline{\lambda} - \lambda} - (\lambda + \overline{\lambda})\frac{h_{21}}{2} = -\frac{1}{2} + \frac{3ia}{2b} - 2ah_{21} = -\frac{1}{2} \tag{10}$$

and the equation reduces to

$$\dot{w} = \lambda w - \frac{1}{2}w^2\overline{w}. \tag{11}$$

The calculations to find the normal form are quite complicated. However, it is a bit easier to find the periodic solution of the equation using perturbations methods. The following MAPLE code does the job.

```
restart;
eq:=omega^2*diff(u(x),x,x) -a*omega*diff(u(x),x)
+omega*diff(u(x),x)*u(x)^2 + u(x);
n:=3;
a:=sum('a||k*eps^k','k'=1..n);
omega:=1+sum('w||k*eps^k','k'=1..n);
u(x):=sum('u||k(x)*eps^k','k'=1..n);
eq:=collect(eq,eps):
for k from 1 to n do
eq||k:=coeff(eq,eps,k):
od;
u1(x):= al*exp(I*x) + bt*exp(-I*x);
u2(x):=0;
w1:=0;
a1:=0;
amp:=coeff(collect(simplify(eq3),exp(I*x)),exp(I*x),1);
```