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BRANCHING OF SOLUTIONS OF NONLINEAR EQUATIONS*

IVAR STAKGOLD†

Abstract. The article is concerned with aspects of the equation $Au = \lambda u$, where λ is a real parameter, u is an element of a real Banach space \mathfrak{B} , and A is a nonlinear operator from \mathfrak{B} into itself such that $A0 = 0$. Unlike the linear case, when the only nontrivial solutions are linear manifolds at particular values of λ , the nonlinear problem may have a very complicated solution set that always includes, however, the solution $u = 0$. Branching theory studies nontrivial solutions $u(\lambda)$ tending to zero as λ tends to some particular value λ_0 , a so-called branch-point.

After some examples and definitions, the connection between branch-points and the eigenvalues of the linearized problem is established through theorems of Leray-Schauder and Krasnoselskii. The construction of branches is described using the methods of Lyapunov-Schmidt and Poincaré-Keller. Three physical examples are then examined in some detail: the Taylor instability for rotating viscous fluids, the nonexistence of buckled states for certain nonlinear rods, and the criticality condition for a simple nuclear reactor.

1. Introduction. Nonlinearity by itself cannot serve as the foundation for a useful mathematical structure. Indeed, finite-dimensional problems already exhibit such a bewildering complexity that no hope can be held out for a comprehensive theory of nonlinear phenomena. Progress is made by settling for something less than complete generality. Global information on the structure of the solution set of a nonlinear problem can usually be obtained only by specifying the nature of the nonlinearity. Here such concepts as convexity, monotonicity, and positivity play their most important role. However, local information regarding solutions in the neighborhood of a known solution can be secured under less restrictive assumptions. This study of local behavior is known as branching theory or bifurcation theory. These synonymous terms are also used, perhaps inappropriately, for recent developments shedding light on some global questions by extending the local analysis in the large. In this category belong many results on positive solutions and on secondary branching.

The present article is of an expository nature; it deals with the basic ideas and techniques of branching theory, and their application to some typical physical problems. Proofs based on topological arguments are omitted, although results are quoted in § 6 and used in § 11. A few historical notes are perhaps in order. The systematic use of nonlinear perturbation theory can be traced to Poincaré's work on celestial mechanics [86]. In the early part of this century Schmidt [96] and Lyapunov [68] spearheaded progress in nonlinear integral equations, Lyapunov's motivation stemming from the study of equilibrium figures for rotating fluids. After a dormant period, the late twenties and early thirties saw a renewal of interest in nonlinear integral equations (Hammerstein [37], Lichtenstein [69], Iglisch [40], Golomb [115]). At about the same time, Hildebrandt and Graves [38] proved the implicit function theorem in Banach space, an essential

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step in the development of branching theory. After World War II, the Soviet school led by Krasnoselskii [63]–[65] and Vainberg [105]–[107] gave the field great impetus. Among the important American contributions in that period are those of Bartle [10], Cronin [27], and Kolodner [61]–[62]. The largest growth has taken place in the sixties; reference to some of the important papers will be made in the main body of the article, but there has been no attempt to make an exhaustive search of the literature.

For the reader interested in further study, two recent books merit special attention: G. H. Pimbley, Jr.: *Eigenfunction Branches of Nonlinear Operators, and their Bifurcation* [85]; *Bifurcation Theory and Nonlinear Eigenvalue Problems*, edited by J. B. Keller and S. Antman [54]. Pimbley's book is devoted principally to mathematical questions, whereas the other is concerned to a much larger extent with physical problems and contains contributions from 14 authors. A number of other pertinent books and review articles are mentioned in the bibliography. The author is pleased to acknowledge his particular debt to the article by Prodi [88].

Despite its role in nonlinear functional analysis, branching theory's principal stimulation arises from physical applications that have proliferated in recent years. Without any claim to completeness, we mention some areas that are drawing the attention of contemporary research workers. In fluid mechanics, problems relating to instabilities of viscous flows are being investigated. A bibliography is given in connection with § 11 which treats the Taylor problem. The buckling of structures is a fertile field for branching theory in solid mechanics. Some references are given in § 12 which deals with the buckling of a rod made of nonlinear material. The von Kármán equations for buckling of plates have been discussed by a number of authors: Friedrichs and Stoker [33], Keller, Keller and Reiss [55], Wolkowisky [110], Berger [14], Berger and Fife [18], Knightly [58], Knightly and Sather [59]. Attention is also drawn to the article by Reiss [92] and the review article by Koiter [60]. Another area of current application is neutron transport theory, a simple example of which is given in § 13. For problems in chemical reactions and nonlinear heat transfer, the reader is referred to the book by Gavalas [33a], and the articles by H. B. Keller and Cohen [46], H. B. Keller [48], Simpson and Cohen [101]. The equations of superconductivity are discussed in Odeh [74], and the Hartree equation by Bazley and Zwahlen [12] and Reeken [91].

The buckling of a thin rod under compression is perhaps the simplest and oldest physical example to illustrate branching. Figure 1 shows a homogeneous,

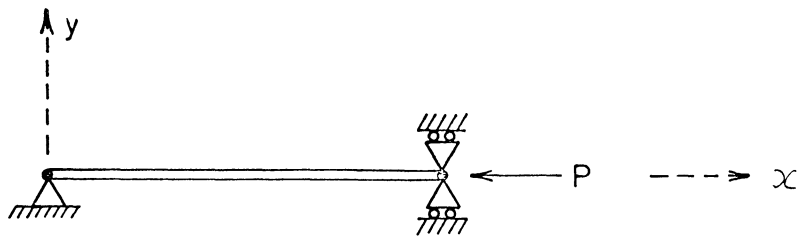


FIG. 1

thin rod whose ends are pinned, the left end being fixed, the right end free to move along the x -axis. In its unloaded state the rod coincides with the portion of the x -axis between 0 and 1. Under a compressive load P , a possible state for the rod is that of pure compression, but experience shows that, for sufficiently large values of P , transverse deflections can occur. Assuming that this buckling takes place in the x, y -plane, we investigate the equilibrium of forces on a portion of the rod including its left end (see Fig. 2). The forces and moments are taken positive as drawn in the figure. Let X be the original x -coordinate of a material point along the rod. This point is moved after buckling to $(X + u, v)$. We let φ be the angle between the tangent to the buckled rod and the x -axis, and s the arclength measured from the left end.

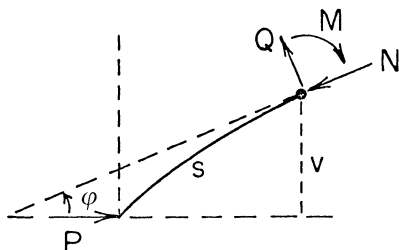


FIG. 2

In § 12, we consider more general constitutive laws, but we restrict ourselves here to the case of an inextensible rod with the *Euler-Bernoulli* law relating the moment M on a cross section and the curvature $d\varphi/ds$. Thus

$$(1.1) \quad s = X$$

and

$$(1.2) \quad M = -EI \frac{d\varphi}{ds},$$

where E and I are given positive, physical constants. These two constitutive equations are supplemented by the geometric relation

$$(1.3) \quad \frac{dv}{ds} = \sin \varphi$$

and the equilibrium condition

$$(1.4) \quad M = Pv.$$

The four equations reduce to the pair of nonlinear differential equations of the first order,

$$(1.5a) \quad \mu v = -\frac{d\varphi}{ds},$$

$$(1.5b) \quad \frac{dv}{ds} = \sin \varphi.$$

where $\mu = P/EI$, and the boundary conditions are

$$(1.5c) \quad v(0) = v(l) = 0.$$

A solution of (1.5) is a triplet (μ, v, φ) ; any solution with $v(s) \not\equiv 0$ is called a *buckled state*. We note that $\mu = 0$ implies $v \equiv 0$ and cannot therefore generate a buckled state. When $\mu \neq 0$, (1.5) is equivalent to the single equation of the second order,

$$(1.6) \quad \frac{d^2 \varphi}{ds^2} + \mu \sin \varphi = 0, \quad 0 < s < l, \quad \varphi'(0) = \varphi'(l) = 0.$$

The lateral deflection $v(s)$ can then be calculated from φ by using (1.5a). One could also derive a single equation for $v(s)$, but the equation is less convenient to handle.

The linearized version of (1.5) for small deflections v and small angles φ is obtained by substituting φ for $\sin \varphi$. Again $\mu = 0$ yields $v \equiv 0$. The linearized problem for v becomes

$$\frac{d^2 v}{ds^2} + \mu v = 0, \quad v(0) = v(l) = 0,$$

which has eigenvalues $n^2 \pi^2 / l^2$, $n = 1, 2, \dots$, with corresponding eigenfunctions $c \sin(n\pi x/l)$. We would like to plot v versus μ , but unfortunately v is itself a function (that is, an infinite-dimensional vector). We therefore content ourselves with a graph of the *maximum deflection* v_{\max} versus $\mu = P/EI$ in Fig. 3.

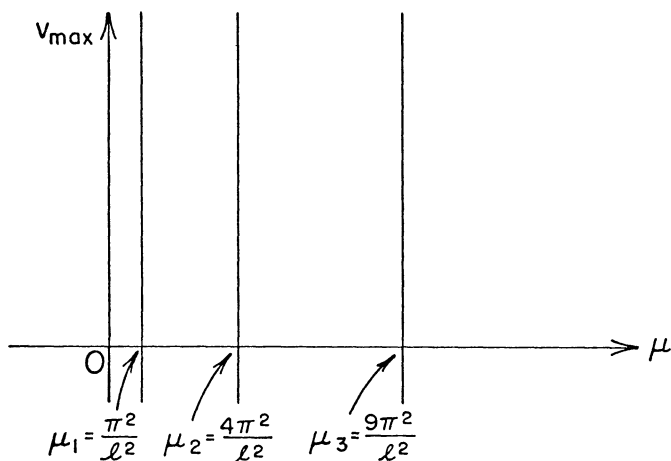


FIG. 3

For loads below the first critical load $P_1 = \pi^2 EI / l^2$, no buckling is possible. At the load P_1 buckling can take place in the mode $c \sin(\pi x/l)$ but the size of the deflection is undetermined owing to the presence of the constant c . As the load slightly exceeds P_1 , the rod must return to its unbuckled state until the second critical load $P_2 = 4\pi^2 EI / l^2$ is reached when buckling can again occur but now

in the new mode $c \sin(2\pi x/l)$, still of undetermined size. Except at the critical loads $P_n = n^2\pi^2 EI/l^2$ there is no buckling. On physical grounds, the picture is clearly unsatisfactory.

The nonlinear problem (1.5) gives a more reasonable prediction of the buckling phenomenon. Again no buckling is possible until the compressive load reaches the first critical load of the linearized theory, but now as this value is exceeded the possible buckling deformation is completely determined except for sign. For values of the load between the first two critical loads there are therefore three possible solutions as shown, in Fig. 4. When P exceeds the second critical

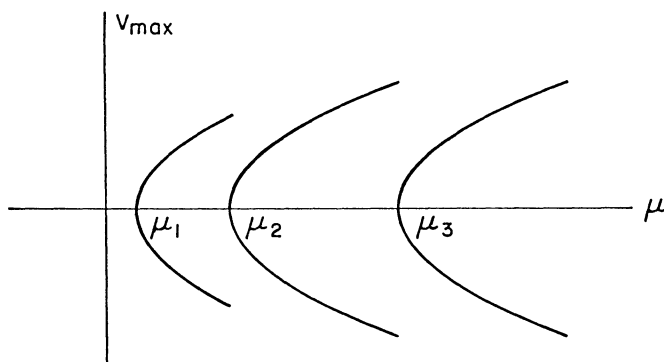


FIG. 4

load a new determinate pair of nontrivial solutions appears, and so forth. The values of μ corresponding to these critical loads are said to be branch-points of the trivial solution because new solutions, initially of small size, appear at these points. Figure 4 (as well as Fig. 3) is called a *branching diagram*.

We shall now justify these statements by exhibiting the closed-form solutions of (1.6). The deflection v can then be obtained from (1.5a). First we observe certain simple properties of the boundary value problem (1.5). If any one of the triplets (μ, v, φ) , $(\mu, v, \varphi + 2n\pi)$, $(\mu, -v, -\varphi)$, $(-\mu, -v, \varphi + \pi + 2n\pi)$ is a solution, so are the other three. All four solutions yield congruent physical deflections. In fact the first two are identical, the third is a reflection of the first about the x -axis; the fourth is a reflection of the first about the origin. In this last case, the reversal in sign of the load accounts for the interchange in the roles of the left and right ends of the rod. Similar remarks apply to (1.6).

All nontrivial lateral deflections $v(s)$ are therefore generated by those solutions of the initial value problem

$$(1.7) \quad \frac{d^2\psi}{ds^2} + \mu \sin \psi = 0, \quad \psi(0) = \alpha, \quad \psi'(0) = 0; \quad \mu > 0, \quad 0 < \alpha < \pi,$$

that happen to have a vanishing derivative at $s = l$. This initial value problem has one and only one solution that can be interpreted as the motion of a simple pendulum, with x being the time and ψ the angle between the pendulum and

the downward vertical. In view of the initial conditions, $\psi(s)$ will be periodic, oscillating about $\psi = 0$, each period consisting of 4 congruent half-bays. After multiplication by $d\psi/ds$, (1.7) becomes a first order equation which can be solved explicitly by elliptic integrals:

$$\psi(s) = 2 \arcsin [k \operatorname{sn}(\sqrt{\mu}s + K)],$$

where

$$k = \sin\left(\frac{\alpha}{2}\right), \quad K = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}.$$

The parameter K takes on values between $\pi/2$ and $+\infty$ and there is a one-to-one correspondence between these values and those of α in $0 < \alpha < \pi$. The period of oscillations of ψ is $4K$. The condition $\psi'(l) = 0$ is satisfied if and only if

$$K = (l/2n)\sqrt{\mu}, \quad n = 1, 2, \dots$$

Since $K > \pi/2$ for nontrivial solutions, these will only be possible for $\mu > \pi^2/l^2$. Given a value of μ exceeding π^2/l^2 there are as many distinct nontrivial solutions $\varphi(s)$ (with $0 < \varphi(0) < \pi$) as there are integers n such that $(l/2n)\sqrt{\mu}$ exceeds $\pi/2$. Thus, we have

$$\begin{aligned} 0 \leq \mu \leq \pi^2/l^2, & \quad \text{only the trivial solution,} \\ \pi^2/l^2 < \mu \leq 4\pi^2/l^2, & \quad \text{one nontrivial solution,} \\ n^2\pi^2/l^2 < \mu \leq (n+1)^2\pi^2/l^2, & \quad n \text{ nontrivial solutions.} \end{aligned}$$

If we then also take into account other values of $\varphi(0)$ and the possibility of negative μ , we obtain Fig. 5 showing the maximum value of $\varphi(s)$ versus μ . The picture is somewhat misleading for the only significant portion is that drawn solid ($\mu > 0$ and $-\pi < \varphi_{\max} < \pi$). The dotted curves for $\mu > 0$ are attributable to angles being determined only modulo 2π . The dotted curves for negative μ just mean that the free end is at $x = -l$. The deflection v is obtained from (1.5a) and yields the branching diagram of Fig. 4, where branches are not extended to infinity because v_{\max} does not grow monotonically (at large loads the rod forms a knot, and further compression tightens the knot so the maximum deflection may decrease although the maximum slope must increase).

2. Definitions and goals. We shall consider the equation

$$(2.1) \quad F(\lambda, u) = 0,$$

where λ is on the real line \mathbb{R} , u is an element of a real Banach space \mathfrak{B} with norm $\|\cdot\|$, and F is a nonlinear transformation from $\mathbb{R} \times \mathfrak{B}$ into \mathfrak{B} . In most physical applications, \mathfrak{B} will be an infinite-dimensional function space. Nevertheless, considerable insight can be derived from the study of finite-dimensional examples. The term “nonlinear eigenvalue problem” is often applied to (2.1) but the description could be misleading. Of principal interest to us is the case where F

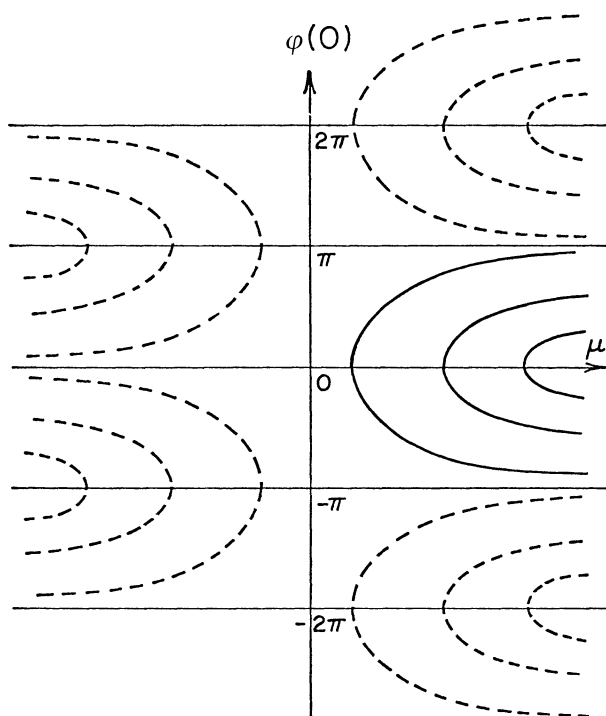


FIG. 5

depends nonlinearly on u and then, unfortunately, most associations brought to mind by spectral terminology are no longer valid.¹

Strictly speaking, a solution of (2.1) is, of course, an ordered pair (λ, u) , but we shall often refer to u itself as the solution (either for fixed λ or depending parametrically on λ). To study branching we must have a simple, explicitly known solution $u(\lambda)$ of (2.1). We shall assume that

$$(2.2) \quad F(\lambda, 0) = 0$$

for all λ , so that $u = 0$ is a solution of (2.1) for all λ . This solution is known as the *basic solution*. Our interest is in studying branching from this basic solution; that is, we want to find solutions of (2.1) which are of small norm.

DEFINITION. We say $\lambda = \lambda^0$ is a *branch-point* of (2.1) if in every neighborhood of $(\lambda^0, 0)$ in $\mathbb{R} \times \mathfrak{B}$, there exists a solution (λ, u) of (2.1) with $\|u\| \neq 0$.

Remark 2.1. A somewhat more precise, but unwieldy, terminology would be that λ^0 is a branch-point of the basic solution of (2.1).

Remark 2.2. The definition places the burden on small neighborhoods of $(\lambda^0, 0)$. Observe that μ_n is a branch-point both in the linear case (Fig. 3) and the nonlinear case (Figs. 4 and 5).

¹ The reader is referred to the article by Friedman and Shinbrot [32] for the case where F is linear in u but nonlinear in λ .

Remark 2.3. Our definition is clearly equivalent to the existence of a sequence (λ^n, u^n) of solutions of (2.1) with $\|u^n\| \neq 0$ such that $(\lambda^n, u^n) \rightarrow (\lambda^0, 0)$.

Remark 2.4. The definition does not guarantee the existence of a continuous branch of solutions. Such a continuous branch will be formed if the boundary of each sufficiently small open ball in $\mathbb{R} \times \mathfrak{B}$ with center at $(\lambda^0, 0)$ contains a solution (λ, u) with $\|u\| \neq 0$.

Remark 2.5. The restriction to real λ and \mathfrak{B} is based on the needs in applications where only real branching is of interest.

Now that we have a definition of a branch-point, we can list some of the principal goals of branching theory:

1. Where are the branch-points? What is the relation of branch-points to the eigenvalues of the linearized problem?
2. How many distinct branches emanate from a branch-point? Is the branching to the left or right?
3. Can we describe the dependence of the branches on λ , at least in the neighborhood of the branch-point?
4. In a physical problem which branch does the physical system follow?
5. How far can branches be extended? When dealing with a function space, can we guarantee that a particular branch represents a positive function? Does secondary branching occur?

The answers to 1–3 are the only ones that fall clearly within the domain of branching theory. Question 4 is related to stability; it is of considerable importance but we shall not have much to say about it. Question 5 deals with global aspects that are not the principal concern of branching theory; nevertheless, some partial answers will be given in some of the examples of physical applications.

In most of what follows we shall assume a particular form of (2.1):

$$(2.3) \quad Au - \lambda u = 0 \quad \text{with } A0 = 0,$$

where A is a nonlinear operator from \mathfrak{B} into \mathfrak{B} . A branch-point of (2.3) is also said to be a branch-point of A .

A preliminary transformation may be required to take a problem as originally stated into (2.3); often this involves passing from a formulation as a differential equation to one as an integral equation.

Although the natural setting for branching problems is in Banach spaces, it is sometimes more convenient to deal with a real Hilbert space whose scalar product will be denoted by $\langle \cdot, \cdot \rangle$.

3. Linearization and the Fréchet derivative. The answer to the first question in the previous section requires a precise notion of linearization. Let A be an operator (possibly nonlinear) from \mathfrak{B} in \mathfrak{B} . We say that A is *linearizable* at u if there exists a bounded linear operator L_u such that

$$(3.1) \quad A(u + h) - Au = L_u h + R_u h \quad \text{with} \quad \lim_{h \rightarrow 0} \frac{\|R_u h\|}{\|h\|} = 0.$$

The operator L_u , linear in h , is said to be the *Fréchet derivative* of A at u , and is also denoted by $A'(u)$. Of course both L_u and R_u vary with u , usually in a nonlinear manner. Formula (3.1) may be regarded as a Taylor expansion in h for

$A(u + h)$, consisting of the 0th order term Au , the linear term $L_u h$ and the remainder $R_u h$ of higher order in h .

Examples and Remarks 3.1. If A is linear, then $A'(u) = A$ independently of u .

3.2. Let A be an operator from \mathfrak{R}_n into \mathfrak{R}_n ; Au is then an n -tuple whose components are nonlinear functions $a_1(u_1, \dots, u_n), \dots, a_n(u_1, \dots, u_n)$, which we assume to be continuously differentiable. We find

$$A(u + h) - Au = L_u h + R_u h,$$

where L_u is an $n \times n$ matrix, with the entries $\partial a_i / \partial u_j$, that is, L_u is the Jacobian evaluated at u and $L_u h$ is a linear transformation acting on the vector h . The remainder $R_u h$ is of higher order in h .

3.3. Consider the nonlinear integral operator

$$Au = \int_0^1 k(x, t) f[u(t)] dt,$$

where $f(u)$ is a differentiable function. Then

$$A'(u)(h) = L_u h = \int_0^1 k(x, t) g[u(t)] h(t) dt,$$

with $g(u) = f'(u)$. Since u is a fixed element in the function space, $L_u h$ is a linear operator on h .

3.4. If A is completely continuous, so is A' . (A nonlinear operator is completely continuous if it is both compact and continuous.)

3.5. A similar definition of derivative can obviously be given for functionals. Let $f(u)$ be a nonlinear functional; we say that f is linearizable at u if there exists a bounded linear functional g_u such that

$$(3.2) \quad f(u + h) - f(u) = g_u(h) + r_u(h) \quad \text{with} \quad \lim_{h \rightarrow 0} \frac{|r_u(h)|}{\|h\|} = 0.$$

We also write $g_u(h) = f'(u)(h)$. A *critical point* for f is a point u at which $f' = 0$.

Suppose we are dealing with a Hilbert space \mathfrak{H} . The Riesz representation theorem guarantees the existence of a unique element z in \mathfrak{H} such that $g_u(h) = \langle z, h \rangle$. Here z is unambiguously determined from u so that $z = Au$, where A is an operator (usually nonlinear) from \mathfrak{H} into \mathfrak{H} . We speak of A as the *gradient* of f ; observe that f is a functional but A is an operator. It is easy to see that this definition agrees with the conventional one in n dimensions. It is useful to proceed in the opposite direction: given a nonlinear operator A , there may exist a functional $f(u)$ such that

$$(3.3) \quad f(u + h) - f(u) = \langle Au, h \rangle + r_u(h),$$

where

$$\lim_{h \rightarrow 0} (|r_u(h)| / \|h\|) = 0.$$

If such a functional f exists, we say that A is a *gradient operator*. For gradient operators the fundamental problem (2.3) is equivalent to finding the critical points of the functional $f(u) - \lambda \|u\|^2 / 2$. The existence of a variational principle raises the hope of a global analysis of the solution of (2.3). In n dimensions, the condition

$\text{curl } A = 0$ is, modulo some technicalities, necessary and sufficient for A to be a gradient. This is equivalent to the Jacobian at every u being symmetric. Similarly, in Hilbert space, an operator A , continuously differentiable in a ball, will be a gradient if and only if $A'(u)$ is symmetric for every u , that is,

$$\langle A'(u)(h_1), h_2 \rangle = \langle h_1, A'(u)(h_2) \rangle$$

for all h_1 and h_2 and every u . If A is a gradient, a scalar potential f can be calculated from

$$f(u) = \int_0^1 \langle A(tu), u \rangle dt;$$

see, for instance, the article by Berger in [54, p. 160].

3.6. A definition of linearization can be given even if the nonlinear operator A is only defined on a domain dense in \mathfrak{B} . The operator L may then be an unbounded operator defined only in a dense domain. This is a useful generalization when dealing with differential equations (see Bazley and Zwahlen [11]).

4. Examples in \mathfrak{R}_1 . We take \mathfrak{B} to be \mathfrak{R}_1 so that u is a real number.

Example 4.1. The most general linear operator is of the form lu . Its only eigenvalue is $\lambda = l$, and u is undetermined. The branching diagram is shown in Fig. 6.

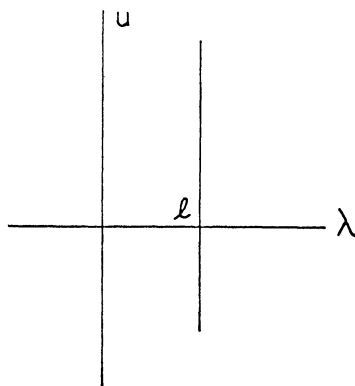


FIG. 6

Example 4.2. Consider the nonlinear operator

$$Au = lu + cu^2, \quad c \neq 0.$$

Equation (2.3) becomes

$$lu + cu^2 = \lambda u$$

which always admits the solution $u = 0$. For $\lambda \neq l$ we also have the solution

$$u = (\lambda - l)/c.$$

The branching diagram is shown in Fig. 7. The only branch-point of the basic solution is at $\lambda = l$. The linearized problem about $u = 0$ is $lu = \lambda u$, which has $\lambda = l$ as its only eigenvalue.

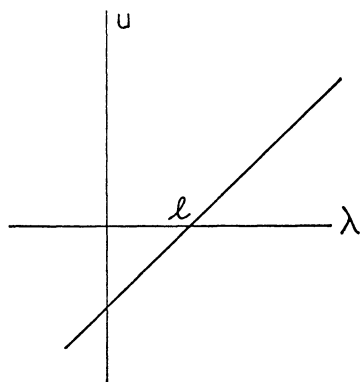


FIG. 7

Example 4.3. Next we look at the nonlinear problem

$$lu + cu^3 = \lambda u, \quad c \neq 0.$$

Again $u = 0$ is a solution for all λ . If $c > 0$, we find two nonzero solutions for $\lambda > l$, given by

$$u = \pm \sqrt{(\lambda - l)/c}.$$

The branching diagram is shown in Fig. 8. Again $\lambda = l$ is a branch-point and $\lambda = l$ is the only eigenvalue of the linearized problem $lu = \lambda u$. If c were negative, the branching would be to the left of $\lambda = l$ instead of to the right.

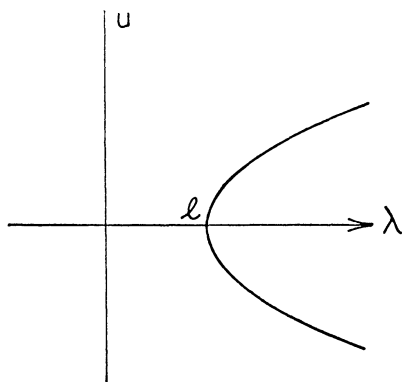


FIG. 8

Example 4.4. Let

$$Au = \begin{cases} u, & |u| < 1, \\ (u^2 + 1)/2, & u > 1, \\ -(u^2 + 1)/2, & u \leq -1. \end{cases}$$

The branching diagram for $Au = \lambda u$ is shown in Fig. 9. Although the problem is nonlinear the branch begins as for a linear problem.

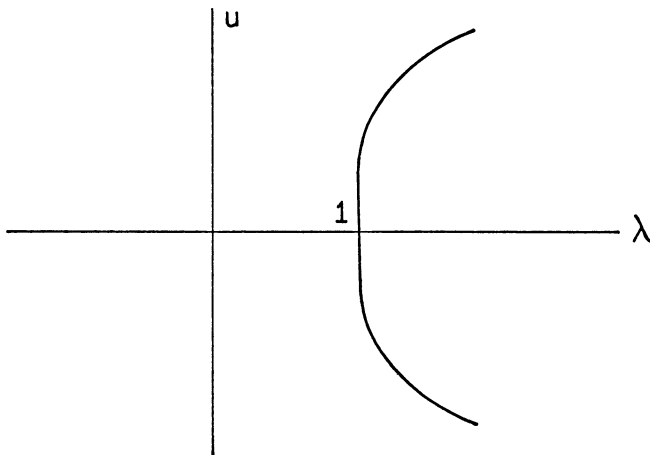


FIG. 9

5. Examples in \mathfrak{R}_2 .

Example 5.1 (*Secondary bifurcation into closed loop*). We look at nonlinear operators of the form

$$Au = Lu + Cu,$$

where

$$Lu = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \beta_1 u_1 \\ \beta_2 u_2 \end{pmatrix}, \quad \beta_2 > \beta_1 > 0,$$

and C is homogeneous of third degree:

$$C(\alpha u) = \alpha^3 Cu.$$

Clearly the linearization of A at $u = 0$ is just L . We take for C the operator

$$Cu = \begin{pmatrix} \gamma_1 u_1(u_1^2 + u_2^2) \\ \gamma_2 u_2(u_1^2 + u_2^2) \end{pmatrix}, \quad \gamma_1 > \gamma_2 > 0.$$

The problem (2.3) becomes

$$(5.1) \quad \beta_1 u_1 + \gamma_1 u_1(u_1^2 + u_2^2) = \lambda u_1, \quad \beta_2 u_2 + \gamma_2 u_2(u_1^2 + u_2^2) = \lambda u_2.$$

The linearized problem has eigenvalues β_1 and β_2 with the respective eigenvectors $\begin{pmatrix} c \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ c \end{pmatrix}$. Let us calculate next the solutions of the nonlinear problem. First, with $u_2 = 0$ and $u_1 \neq 0$, we have

$$u_1 = \pm \sqrt{(\lambda - \beta_1)/\gamma_1},$$

so that there is branching to the right from $\lambda = \beta_1$ and the branch is a parabola in the u_1, λ -plane. With $u_2 \neq 0$ and $u_1 = 0$, we find

$$u_2 = \pm \sqrt{(\lambda - \beta_2)/\gamma_2}$$

which branches to the right from $\lambda = \beta_2$; the branch is a parabola in the u_2, λ -plane. Finally we look for solutions with neither u_1 nor u_2 vanishing. Then we have simultaneously

$$u_1^2 + u_2^2 = (\lambda - \beta_1)/\gamma_1 \quad \text{and} \quad u_1^2 + u_2^2 = (\lambda - \beta_2)/\gamma_2,$$

so that

$$\lambda = \lambda^* = (\beta_2\gamma_1 - \beta_1\gamma_2)/(\gamma_1 - \gamma_2) > \beta_2$$

and

$$u_1^2 + u_2^2 = (\beta_2 - \beta_1)/(\gamma_1 - \gamma_2) > 0.$$

This particular solution to the nonlinear problem (5.1) is a circle lying in the plane $\lambda = \lambda^*$. At $\lambda = \lambda^*$, we find that the two earlier parabolas are equidistant from the λ -axis, the square of the common distance being just $(\beta_2 - \beta_1)/(\gamma_1 - \gamma_2)$. Thus the circle is connected to the parabolas as shown in Fig. 10. The circle may be regarded as a secondary branching from either of the two nonintersecting parabolas. This

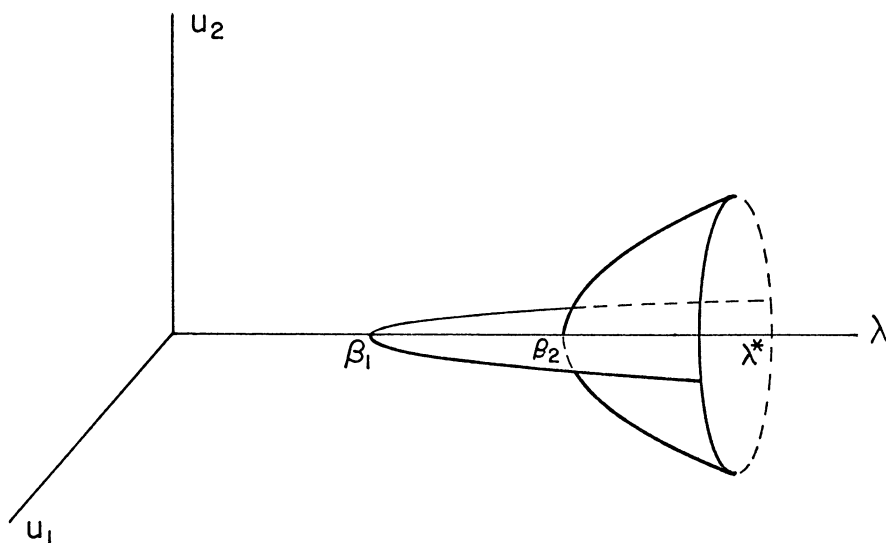


FIG. 10

particularly simple example gives some hint as to the possible variety in branch structure. If we linearize (5.1) about the solution represented, say, by one-half of the first parabola, it must follow that the linearized problem has an eigenvalue at λ^* . This can be checked by a straightforward but slightly tedious calculation. The type of secondary bifurcation illustrated here is unusual; instead there is usually a “twig” growing some place along one of the two main branches without being connected to the other branch. An example given by Pimbley [85] is of this type:

$$\lambda u_1 = \beta_1 u_1 + \frac{3}{4}\beta_1 u_1^3 + \frac{3}{2}\beta_1 u_1 u_2^2, \quad \lambda u_2 = \beta_2 u_2 + \frac{3}{4}\beta_2 u_2^3 + \frac{3}{2}\beta_2 u_1^2 u_2,$$

with $\beta_2/\beta_1 > 1/2$.

Example 5.2 (Typical branching from a multiple eigenvalue of the linearized problem). Consider the operator of Example 5.1 with $\beta_1 = \beta_2 = \beta$. Our previous calculations are unchanged but now there is no secondary branching. Both main branches are parabolas starting at $\lambda = \beta$. In this typical case two branches emanate from $\lambda = \beta$ which is an eigenvalue of multiplicity two of the linearized problem.

Example 5.3 (A continuum of branches emanating from a multiple eigenvalue). In the operator of Example 5.1 set $\beta_1 = \beta_2 = \beta$, $\gamma_1 = \gamma_2 = \gamma > 0$. The nonlinear problem (5.1) reduces to the single equation $u_1^2 + u_2^2 = (\lambda - \beta)/\gamma$. These solutions branch from $\lambda = \beta$ and form the surface of a paraboloid of revolution about the λ -axis as in Fig. 11. Observe that $\lambda = \beta$ is an eigenvalue of multiplicity 2 of the linearized problem, but the number of branches of the nonlinear problem greatly exceeds 2. We note for further reference that the operator in our case is an odd gradient operator, that is, $A(-u) = -Au$. A branching diagram using $\|u\|$ as dependent variable would have completely masked the interesting nature of the branching.

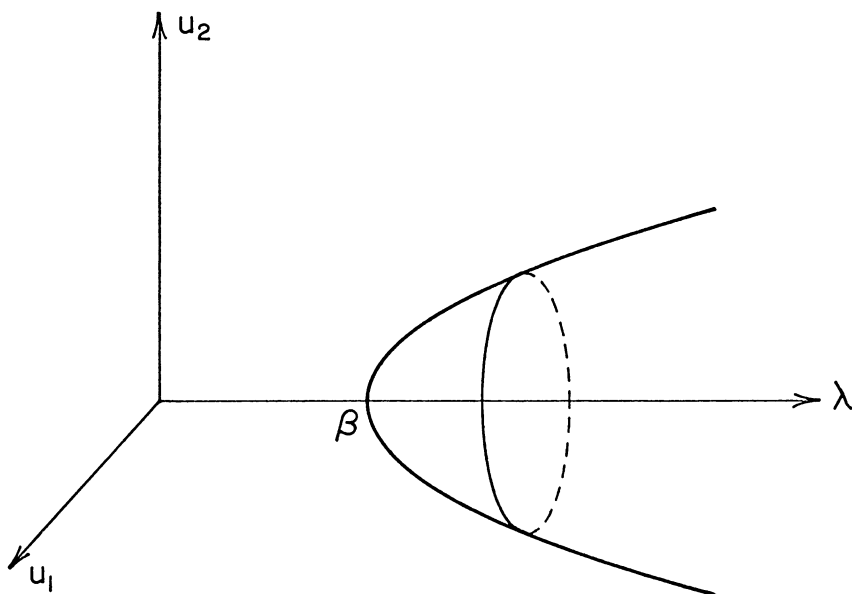


FIG. 11

Example 5.4 (A single branch emanating from a multiple eigenvalue). Consider the odd gradient operator defined by

$$A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + 2u_1u_2 \\ u_2 + u_1^2 + 2u_2^2 \end{pmatrix}.$$

Clearly $\lambda = 1$ is the only eigenvalue of the linearized problem and has multiplicity 2. The nonlinear problem $Au = \lambda u$ has only the solution $u_1 = 0$, $u_2 = (\lambda - 1)/2$ in addition to the trivial solution. Thus a single branch emanates from $\lambda = 1$.

Example 5.5 (No branching from an eigenvalue). Let

$$Au = \begin{pmatrix} u_1 + u_2^3 \\ u_2 - u_1^3 \end{pmatrix}.$$

The nonlinear problem (2.3) becomes

$$u_1 + u_2^3 = \lambda u_1, \quad u_2 - u_1^3 = \lambda u_2.$$

Multiplying these equations by u_2 and u_1 , respectively, and subtracting, we find

$$u_1^4 + u_2^4 = 0,$$

so that $u_1 = u_2 = 0$. Thus the only solution of $Au = \lambda u$ is the basic solution $u = 0$ and there is no branching. The derivative of A at $u = 0$ is the identity operator so that the linearized problem has the eigenvalue $\lambda = 1$ with multiplicity 2. An example of no branching from a geometrically simple eigenvalue is given in Remark 6.3.

Remark 5.1. The two-dimensional examples just presented can be translated into the language of integral equations by using appropriate degenerate kernels. For instance, the picturesque example (5.1) is equivalent to the integral equation

$$\lambda u(x) = \int_{-\pi}^{\pi} \{ [p_1(x)p_1(t) + p_2(x)p_2(t)]u(t) + [p_1(x)q_1(t) + p_2(x)q_2(t)]u^3(t) \} dt, \\ -\pi < x < \pi,$$

where

$$p_1(x) = (\beta_1/2\pi)^{1/2}, \quad p_2(x) = (\beta_2/\pi)^{1/2} \sin x,$$

and $q_1(x)$, $q_2(x)$ are even and odd, respectively, and satisfy

$$\int_{-\pi}^{\pi} q_1 p_1^3 dx = 3 \int_{-\pi}^{\pi} p_1 p_2^2 q_1 dx = \gamma_1$$

and

$$\int_{-\pi}^{\pi} q_2 p_2^3 dx = 3 \int_{-\pi}^{\pi} p_1^2 p_2 q_2 dx = \gamma_2.$$

6. Basic theorems of branching. Let A be an operator with $A0 = 0$; assume that $A'(0) = L$ exists so that

$$(6.1) \quad Ah = Lh + Rh \quad \text{with} \quad \lim_{h \rightarrow 0} \frac{\|Rh\|}{\|h\|} = 0.$$

We investigate branching from the basic solution for the equation

$$(6.2) \quad Au - \lambda u = 0.$$

THEOREM 1. *The number λ^0 can be a branch-point of A only if it is in the spectrum of L .*

Proof. Let λ^0 be a branch-point. There must therefore exist a sequence $(\lambda^n, u^n) \rightarrow (\lambda^0, 0)$ with $\|u^n\| \neq 0$ and $Au^n - \lambda^n u^n = 0$, or, using (6.1),

$$Lu^n - \lambda^0 u^n \equiv (\lambda^n - \lambda^0)u^n - Ru^n.$$

Setting $z^n = u^n/\|u^n\|$, we find $Lz^n - \lambda^0 z^n = f^n$, where $\lim_{n \rightarrow \infty} \|f^n\| = 0$. If $L - \lambda^0 I$ had a bounded inverse, we would have

$$\|z^n\| \leq \|(L - \lambda^0 I)^{-1}\| \|f^n\|,$$

which is impossible since $\|z^n\| = 1$ and $\|f^n\| \rightarrow 0$. Therefore the branch-point λ^0 must be in the spectrum of the linearized problem.

Remark 6.1. The theorem tells us that all branch-points of the nonlinear problem are found in the spectrum of the linearized problem, but as we have seen in Example 5.5, not every point of the spectrum generates a branch-point. Next we state some theorems that guarantee that points in the spectrum of the linearized problem in fact generate branches for the nonlinear problem.

THEOREM 2 (Leray–Schauder). *If A is completely continuous, and $\lambda^0 \neq 0$ is an eigenvalue of odd multiplicity for L , then λ^0 is a branch-point of the basic solution of the nonlinear problem.*

Remark 6.2. The proof is based on topological degree theory and is omitted (see Krasnoselskii [65]). The nature of the proof is not constructive so that little information is obtained about the structure of the branch. The theorem does not contradict Example 5.5, for there the multiplicity of the eigenvalue is 2. In applying Theorem 2, one is limited by the need for predicting the multiplicity of the eigenvalues for a linear problem. The only available theorems are those which state that certain eigenvalues are simple. This is the case, for instance, for Sturm–Liouville problems for ordinary differential equations, for oscillatory matrices and kernels, and for the lowest eigenvalue of elliptic boundary value problems.

Remark 6.3. The multiplicity of λ^0 is the dimension of the union of the null spaces of $(L - \lambda^0 I)^m$, $m = 1, 2, \dots$. The Riesz theory guarantees that each null space is strictly contained in the succeeding one until a certain index p is reached; thereafter all the null spaces have the same dimension, necessarily finite (see [118]). This notion of multiplicity reduces to that of algebraic multiplicity in finite-dimensional problems. For symmetric problems, $p = 1$, and algebraic multiplicity coincides with geometric multiplicity. The example in $R^2: (u_1, u_2) \rightarrow (u_1^3 + u_2, u_1^2 u_2 - u_1^3)$ shows that a geometrically simple eigenvalue may not generate a branch.

Remark 6.4. In [26], Crandall and Rabinowitz have ingeniously applied degree theory to the study of nonlinear Sturm–Liouville problems.

THEOREM 3 (Krasnoselskii). *Let A be a completely continuous operator which is the gradient of a uniformly differentiable functional, and let A have a second derivative at the origin. Then any nonzero eigenvalue of L is a branch-point of A .*

Remark 6.5. The proof, which is omitted, is based on the category theory of Liusternik and Schnirelmann. For a generalization, see Browder [120].

Remark 6.6. The operator in Example 5.5 is not a gradient operator so that no contradiction arises.

Remark 6.7. Berger [13], [15] has announced the following improvement of Theorem 3: For an odd gradient operator, the number of branches emanating from the branch-point is greater than or equal to the multiplicity of the eigenvalue. Example 5.3 shows the need for the “greater than or equal” part of the theorem. Example 5.4 is not odd, and Example 5.5 is not a gradient operator.

7. Pseudoinverse. Let L be a completely continuous linear operator and λ^0 one of its nonzero *real* eigenvalues. The null space \mathfrak{N} of $L - \lambda^0 I$ is of finite dimension, say k . The range \mathfrak{R} of $L - \lambda^0 I$ is of codimension k but may not be complementary to \mathfrak{N} (we say that 2 subspaces are complementary if every element of \mathfrak{H} has an unambiguous decomposition as the sum of two elements, one in each subspace). Let us choose in a definite manner 2 subspaces \mathfrak{B} and \mathfrak{Z} complementary to \mathfrak{N} and \mathfrak{R} , respectively. For instance, we could choose $\mathfrak{B} = \mathfrak{N}^\perp$ and $\mathfrak{Z} = \mathfrak{R}^\perp$, the orthogonal complements of \mathfrak{N} and \mathfrak{R} , but this choice is not always best for our purposes.

By definition, the equation

$$(7.1) \quad Lu - \lambda^0 u = f$$

has solution(s) if and only if f belongs to \mathfrak{R} . As is well known, $\mathfrak{R} = (\mathfrak{N}^*)^\perp$, where \mathfrak{N}^* is the null space, necessarily of dimension k , of the adjoint $\mathfrak{L}^* - \lambda^0 I$. If f belongs to \mathfrak{R} , equation (7.1) has many solutions, any two of which differ by an element of \mathfrak{N} , but there is one and only one solution w in \mathfrak{B} . The mapping from f to w , when extended to the whole space by setting $w = 0$ for f in \mathfrak{Z} , is known as a *pseudoinverse* of $L - \lambda^0 I$. This pseudoinverse is a bounded transformation which depends on the choices of \mathfrak{Z} and \mathfrak{B} , but when no confusion arises the dependence is suppressed. Thus

$$(7.2) \quad Tf = \begin{cases} 0, & f \in \mathfrak{Z}, \\ w, & f \in \mathfrak{R}, \end{cases}$$

where w is the one and only solution in \mathfrak{B} of $Lu - \lambda^0 u = f$. For arbitrary f , we write $f = f_1 + f_2$, where f_1 belongs to \mathfrak{Z} and f_2 to \mathfrak{R} , and define $Tf = Tf_2$.

Special cases.

Case 7.1 (Riesz index 1). If the null spaces of $L - \lambda^0 I$ and $(L - \lambda^0 I)^2$ coincide, then \mathfrak{N} and \mathfrak{R} are complementary subspaces, although not necessarily orthogonal. This covers all symmetric problems and some nonsymmetric ones. We then choose $\mathfrak{Z} = \mathfrak{R}$, $\mathfrak{B} = \mathfrak{N}$, so that

$$(7.3) \quad Tf = \begin{cases} 0, & f \in \mathfrak{N}, \\ w, & f \in \mathfrak{R}, \end{cases}$$

where w is the one and only solution in \mathfrak{N} of $Lu - \lambda^0 u = f$.

Case 7.2. A further specialization of Case 7.1 occurs when L is self-adjoint; we then have $\mathfrak{N} = \mathfrak{N}^\perp$ and we may (and do) choose $\mathfrak{J} = \mathfrak{N}$, $\mathfrak{B} = \mathfrak{N}^\perp$. Thus

$$(7.4) \quad Tf = \begin{cases} w, & f \in \mathfrak{N}^\perp, \\ 0, & f \in \mathfrak{N}, \end{cases}$$

where w is the one and only solution in \mathfrak{N}^\perp of $Lu - \lambda^0 u = f$.

8. Linear perturbation theory. Let $L(\varepsilon)$ be a linear operator depending “smoothly” on the parameter ε , at least for $|\varepsilon|$ sufficiently small. For small ε , we want to relate the eigenvalues and eigenfunctions of

$$(8.1) \quad L(\varepsilon)u = \lambda u$$

to those, presumably known, of $L = L(0)$.

We shall assume that $L(\varepsilon)$ is completely continuous and symmetric for every ε in a neighborhood of 0, that $\lambda^0 \neq 0$ is an eigenvalue (necessarily isolated) of L . The null space \mathfrak{N} of $L - \lambda^0 I$ is of dimension k and has the orthonormal basis $\varphi_1, \dots, \varphi_k$. We want to find out what happens to λ^0 under small perturbations. Our treatment follows the one in Rellich [93].

THEOREM 4. *For small ε , there exist k numbers $\lambda_1(\varepsilon), \dots, \lambda_k(\varepsilon)$ and k elements $\varphi_1(\varepsilon), \dots, \varphi_k(\varepsilon)$, orthonormal for real ε , satisfying*

$$L(\varepsilon)\varphi_i(\varepsilon) = \lambda_i(\varepsilon)\varphi_i(\varepsilon), \quad \lim_{\varepsilon \rightarrow 0} \lambda_i(\varepsilon) = \lambda^0, \quad \lim_{\varepsilon \rightarrow 0} \text{span} [\varphi_1(\varepsilon), \dots, \varphi_k(\varepsilon)] = \mathfrak{N}.$$

We shall only sketch the principal constructive ideas of the proof, stressing the parts that will apply later to nonlinear investigations. Problem (8.1) can be rewritten as

$$(8.2) \quad Lu - \lambda^0 u = \delta u - R(\varepsilon)u, \quad \delta = \lambda - \lambda^0, \quad R(\varepsilon) = L(\varepsilon) - L.$$

If (8.2) has a solution u , the right side must be in the range of $L - \lambda^0 I$, so that

$$(8.3) \quad \langle \delta u - R(\varepsilon)u, \varphi_i \rangle = 0, \quad i = 1, \dots, k.$$

Any element u in \mathfrak{H} can be unambiguously decomposed as

$$(8.4) \quad u = v + w, \quad v = \sum_{i=1}^k c_i \varphi_i, \quad \langle w, \varphi_i \rangle = 0, \quad i = 1, \dots, k.$$

Keeping in mind condition (8.3), and using (8.2) and (7.4), we find

$$(8.5) \quad u = v + T[\delta I - R(\varepsilon)]u,$$

where T is the pseudoinverse of $L - \lambda^0 I$.

Equation (8.1) is equivalent to the pair (8.3) and (8.5), to which we now turn our attention. With v a fixed element of \mathfrak{N} and with δ, ε fixed and sufficiently small, equation (8.5) has a unique solution $u = \bar{u}(c_1, \dots, c_k, \delta, \varepsilon)$ or $\bar{u}(v, \delta, \varepsilon)$. This is a consequence of the fact that the right side of (8.5), regarded as a transformation from

$u \in \mathfrak{H}$ into \mathfrak{H} , is continuous for all δ and ε and is a contraction for any δ and ε such that

$$|\delta| + \|R(\varepsilon)\| < \frac{1}{\|T\|}.$$

This unique solution \bar{u} of (8.5) is linear in v , and can be obtained by successive approximations as

$$(8.6) \quad \bar{u} = \sum_{m=0}^{\infty} \{T[\delta I - R(\varepsilon)]\}^m v.$$

Thus

$$\bar{u} = \sum_{i=1}^k c_i S \varphi_i,$$

where S is a linear operator from \mathfrak{H} into \mathfrak{H} depending on the parameters ε and δ :

$$S \varphi_i = S(\delta, \varepsilon) \varphi_i = \sum_{m=0}^{\infty} \{T[\delta I - R(\varepsilon)]\}^m \varphi_i.$$

Observe that $S(\delta, 0) = I$.

Substituting \bar{u} in (8.3) we find

$$\langle \bar{u}, [\delta I - R(\varepsilon)] \varphi_i \rangle = 0, \quad i = 1, \dots, k,$$

or

$$\sum_{j=1}^k c_j \langle S \varphi_j, [\delta I - R(\varepsilon)] \varphi_i \rangle = 0, \quad i = 1, \dots, k,$$

which is a set of k equations in the $k + 2$ numerical variables $c_1, \dots, c_k, \varepsilon, \delta$. Note that the equations are linear in c_1, \dots, c_k , but not in ε and δ . If we add a normalization condition on the coefficients c_1, \dots, c_k , we can hope to express them and δ in terms of ε . To find δ in terms of ε is to determine the perturbed eigenvalues since $\lambda(\varepsilon) = \lambda^0 + \delta(\varepsilon)$. Unlike the nonlinear perturbation problem to be studied later, the perturbed eigenvalues are characterized independently of the eigenfunctions by

$$(8.7) \quad \det(\langle S \varphi_j, [\delta I - R(\varepsilon)] \varphi_i \rangle) = 0.$$

The first order change in the eigenvalues is obtained by setting $S = I$. If λ^0 is a simple eigenvalue we find

$$\delta = \lambda(\varepsilon) - \lambda^0 \sim \langle \varphi, R(\varepsilon) \varphi \rangle,$$

a result familiar to all physicists.

Consider the case where $L(\varepsilon)$ is a power series in ε for small ε .² Then $S(\varepsilon, \delta)$ is a power series in ε and δ for small ε and δ , and so is the $k \times k$ determinant (8.7). Setting this determinant equal to $F(\delta, \varepsilon)$, we have the equation

$$F(\delta, \varepsilon) = 0,$$

where $F(\delta, 0) = \delta^k$ since the determinant is then diagonal.

² We say $L(\varepsilon)$ is a power series in ε , if for each $u \in \mathfrak{H}$, $L(\varepsilon)u$ is a power series in ε which converges in the norm of \mathfrak{H} .

According to the Weierstrass preparation theorem (see [93] and [20]), we can write

$$F(\delta, \varepsilon) = [\delta^k + a_1(\varepsilon)\delta^{k-1} + \cdots + a_k(\varepsilon)]E(\delta, \varepsilon),$$

where a_1, \dots, a_k, E are power series near the origin and $E(0, 0) = 1$, $\lim_{\varepsilon \rightarrow 0} a_i(\varepsilon) = 0$. There exist therefore k functions $\delta_1(\varepsilon), \dots, \delta_k(\varepsilon)$ such that

$$\delta^k + a_1(\varepsilon)\delta^{k-1} + \cdots + a_k(\varepsilon) = [\delta - \delta_1(\varepsilon)] \cdots [\delta - \delta_k(\varepsilon)]$$

and

$$\lim_{\varepsilon \rightarrow 0} \delta_i(\varepsilon) = 0.$$

Since the eigenvalues of $L(\varepsilon)$ are real, $\lambda^0 + \delta_i(\varepsilon)$, which is an eigenvalue of L for ε small, must be real for ε small. We thus have established the existence of k real “branches” of eigenvalues. One can also show fairly easily that there are k associated eigenfunction branches $\varphi_1(\varepsilon), \dots, \varphi_k(\varepsilon)$ that form an orthonormal set for real ε .

If $L(\varepsilon)$ is not analytic in ε , the analysis is somewhat more difficult and we cannot expect analytic dependence of λ on ε . We discuss some aspects of this question in the next section.

9. Nonlinear perturbation theory by the Lyapunov–Schmidt method. We return to the nonlinear problem (6.2), repeated here for convenience,

$$(9.1) \quad Au - \lambda u = 0, \quad A0 = 0.$$

We assume that $A'(0) = L$ is completely continuous. From (6.1), we have

$$Au = Lu + Ru \quad \text{with} \quad \lim_{\|u\| \rightarrow 0} \frac{\|Ru\|}{\|u\|} = 0.$$

The theorems of § 6 tell us a great deal about when branching can occur. We now try to get more detailed information about the number and initial shape of the branches. Let $\lambda^0 \neq 0$ be a real eigenvalue of L . We want to investigate branching from λ^0 . To this end we set $\delta = \lambda - \lambda^0$ so that (9.1) becomes

$$(9.2) \quad Lu - \lambda^0 u = \delta u - Ru.$$

Although we are interested in solutions u of small norm with δ small, our preliminary transformations will be valid without restriction. Let \mathfrak{N} be the null space (of dimension k , say) of $L - \lambda^0 I$, and let \mathfrak{B} be a fixed subspace complementary to \mathfrak{N} . We choose a real orthonormal basis $\{\varphi_1, \dots, \varphi_k\}$ for \mathfrak{N} . Any element u of \mathfrak{H} has a unique decomposition

$$(9.3) \quad u = v + w = Pu + Qu = \sum_{i=1}^k c_i \varphi_i + Qu,$$

where P and Q are complementary projections on \mathfrak{N} and \mathfrak{B} .

Equation (9.2) has solutions only if $\delta u - Ru$ is in the range \mathfrak{R} of $L - \lambda^0 I$, so that the consistency conditions

$$(9.4) \quad \langle \delta(v + w) - R(v + w), \varphi_i^* \rangle = 0, \quad i = 1, \dots, k,$$

must hold. Here $\varphi_1^*, \dots, \varphi_k^*$ is a basis for the null space \mathfrak{N}^* of the adjoint $L^* - \lambda^0 I$. Since $Lv - \lambda^0 v = 0$, (9.2) becomes

$$(9.5) \quad Lw - \lambda^0 w = \delta(v + w) - R(v + w).$$

Thus any solution (δ, u) of (9.2) must satisfy (9.4) and (9.5) simultaneously, where we have set $v = Pu$, $w = Qu$. Conversely, if v in \mathfrak{N} and w in \mathfrak{W} satisfy (9.4) and (9.5), then $\delta(v + w) - R(v + w)$ belongs to \mathfrak{R} and $u = v + w$ satisfies $Lu - \lambda^0 u = \delta u - Ru$. Therefore (9.2), and hence (9.1), is equivalent to (9.4) and (9.5) with v in \mathfrak{N} and w in \mathfrak{W} . Henceforth we study the pair (9.4) and (9.5).

In view of (9.4), equation (9.5) is consistent and is therefore equivalent to

$$(9.6) \quad w = T[\delta(v + w) - R(v + w)],$$

where T is the pseudoinverse of $L - \lambda^0 I$ defined in (7.2). Observe that any solution of (9.6) automatically belongs to \mathfrak{W} . Problem (9.1) is therefore equivalent to finding a real number δ , an element v in \mathfrak{N} and an element w satisfying the pair

$$(9.7) \quad \begin{aligned} \langle (\delta I - R)(v + w), \varphi_i^* \rangle &= 0, & i = 1, \dots, k, \\ w &= T(\delta I - R)(v + w). \end{aligned}$$

Of course any element v in \mathfrak{N} is of the form $\sum_{i=1}^k c_i \varphi_i$, so that we are looking for real numbers δ, c_1, \dots, c_k and an element w satisfying

$$(9.8a) \quad \left\langle (\delta I - R) \left(w + \sum_{j=1}^k c_j \varphi_j \right), \varphi_i^* \right\rangle = 0, \quad i = 1, \dots, k,$$

$$(9.8b) \quad w = T(\delta I - R) \left(\sum_{j=1}^k c_j \varphi_j + w \right).$$

The appearance of (9.8) is very similar to (8.3) and (8.5) for a linear problem. We shall have occasion to comment from time to time on this analogy.

So far we have not said anything about “small” solutions. Let us now turn to the branching problem where δ , v and w are all small. We propose the following program. Let δ be a fixed number and v a fixed element of \mathfrak{N} , with $|\delta|$ and $\|v\|$ sufficiently small. It will be shown that under mild conditions on R (or equivalently, on A), equation (9.8b) has one and only one small solution $w = \bar{w}(\delta, v)$ or $w = \bar{w}(\delta, c_1, \dots, c_k)$. Moreover \bar{w} depends continuously on its arguments and can be constructed by the standard iterative procedure. We then substitute \bar{w} into the *branching equation* (9.8a) to obtain k nonlinear equations in the $k + 1$ numbers δ, c_1, \dots, c_k . Thus the original problem is reduced to a finite-dimensional problem that, unfortunately, can still be cumbersome. We want to predict the number of solutions (which can range from 0 to infinity!) and perhaps express c_1, \dots, c_k in

terms of δ .³ Quantitative information requires more knowledge about \bar{w} than its mere existence. Under more detailed assumptions on R , the needed asymptotic behavior of \bar{w} for small arguments can be obtained.

The difference between nonlinear branching theory and linear perturbation theory can now be pointed out. In linear perturbation there is a natural small parameter ε in the operator $L(\varepsilon)$. In the nonlinear case the nearest equivalent is $\|u\|$, but this is not entirely satisfactory. The right side of (9.8b) is a contraction mapping only for small c_1, \dots, c_k, δ , and w , whereas the right side of (8.5) is a contraction for small δ and ε , independently of c_1, \dots, c_k .

To carry out the nonlinear program, we want to first analyze the existence and uniqueness of small solutions of (9.8b). We need the following hypothesis.

(H₁) A is strictly differentiable at 0.

By definition, A is strictly differentiable at a if

$$\|A(a + h_1) - A(a + h_2) - L_a(h_1 - h_2)\| = K(a, h_1, h_2)\|h_1 - h_2\|,$$

where $K(a, h_1, h_2) \rightarrow 0$ as $\|h_1\| \rightarrow 0$ and $\|h_2\| \rightarrow 0$. Thus the remainder $R_a h$ satisfies the condition

$$\|R_a h_1 - R_a h_2\| = K(a, h_1, h_2)\|h_1 - h_2\|.$$

It is easy to show that if A is continuously differentiable at a , then it is strictly differentiable (see [22]). Strict differentiability therefore lies somewhere between differentiability and continuous differentiability.

THEOREM 5. *Under (H₁), there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ such that $|\delta| \leq \delta_0$ and $\|v\| \leq \varepsilon_0$ imply that equation (9.8b) has one and only one solution $w = \bar{w}(\delta, v)$ in the ball $\|w\| \leq \varepsilon_0$. Moreover \bar{w} depends continuously on its arguments and $\|\bar{w}\| \leq \|v\|$.*

Proof. This is a straightforward application of the contraction mapping principle. Denote by Bw the mapping $T[\delta(v + w) - R(v + w)]$ regarded as a function of w , with δ and v as parameters. Then by the strict differentiability,

$$(9.9) \quad \|Bw\| \leq \|T\|(|\delta| + |K|)(\|v\| + \|w\|),$$

where $K \rightarrow 0$ as $\|v\|, \|w\| \rightarrow 0$. Choose $\delta_0 = 1/(4\|T\|)$ and ε_1 such that $\|v\| \leq \varepsilon_1$ and $\|w\| \leq \varepsilon_1$ imply $|K| \leq 1/(4\|T\|)$. Then $\|Bw\| \leq \varepsilon_1$ so that the ball $\|w\| \leq \varepsilon_1$ is mapped into itself whenever $\|v\| \leq \varepsilon_1$ and $|\delta| \leq \delta_0$. To show that we have a contraction, note that

$$\|Bw_1 - Bw_2\| \leq \|T\|(|\delta| + |K'|)(\|w_1 - w_2\|),$$

where $|K'| \rightarrow 0$ as $v, w_1, w_2 \rightarrow 0$. Thus we can choose ε_2 such that $\|v\| \leq \varepsilon_2$, $\|w_1\| \leq \varepsilon_2$, $\|w_2\| \leq \varepsilon_2$ imply $|K'| \leq 1/(4\|T\|)$. Then with $|\delta| \leq \delta_0$, $\|Bw_1 - Bw_2\| \leq \frac{1}{2}\|w_1 - w_2\|$. If ε_0 is the lesser of ε_1 and ε_2 , then for any δ with $|\delta| \leq \delta_0$ and any v with $\|v\| \leq \varepsilon_0$, the mapping B is a contraction of the ball $\|w\| \leq \varepsilon_0$ into itself. It has a unique fixed point $w = \bar{w}(\delta, v)$ which satisfies $\|\bar{w}\| \leq \|v\|$ uniformly for $|\delta| \leq \delta_0$. Clearly the fixed point depends continuously on δ and v .

³ Admittedly this is not always possible, for instance, when A is linear or when A is nonlinear with a branch that is initially perpendicular to the λ -axis (see Example 4.4; similar phenomena also arise in differential equations or integral equations [111]).

Next we wish to substitute \bar{w} in (9.8a) and look for solutions. The general case is quite complicated and even then progress is possible only by developing A in a Taylor expansion to more terms. Fairly complete treatments can be found in Pimbley [85] and Sather [94]. Instead we make a hypothesis which greatly simplifies the basic equation (9.8).

(H₂) The Riesz index of $L - \lambda^0 I$ is unity.

Now the operators $(L - \lambda^0 I)^m$, $m = 1, 2, \dots$, all have the same null space (of dimension k , the multiplicity). We may therefore use the pseudoinverse defined in (7.3), and (9.8b) becomes

$$(9.10) \quad w = \delta T w - TR \left(\sum_{j=1}^k c_j \varphi_j + w \right),$$

whose fixed point \bar{w} can now be shown to satisfy

$$(9.11) \quad \bar{w} = o(\|v\|),$$

uniformly in $|\delta| \leq \delta_0$. The second simplification occurs in (9.8a), which states that the element $(\delta I - R)(v + w)$ belongs to \mathfrak{R} . Since \mathfrak{R} is complementary to \mathfrak{R} , the projection of the element on \mathfrak{R} vanishes, that is,

$$(9.12) \quad \delta \sum_{j=1}^k c_j \varphi_j = PR \left(\sum_{j=1}^k c_j \varphi_j + \bar{w} \right).$$

THEOREM 6. *Let (H₁) and (H₂) hold and let λ^0 be a simple eigenvalue of L . Thus, \mathfrak{R} is one-dimensional and we let φ be a real normalized eigenelement. Then λ^0 is a branch-point of A .*

Proof. Set $v = c\varphi$, $\|v\| = |c|$. Equation (9.12) becomes

$$\delta c \varphi = PR[c\varphi + \bar{w}(\delta, c)].$$

The right side is proportional to φ , equal, say, to $\gamma(\delta, c)\varphi$. It remains to show that

$$\delta c = \gamma(\delta, c)$$

has a nontrivial solution. From (9.11), it follows that

$$\lim_{c \rightarrow 0} \frac{\gamma(\delta, c)}{c} = 0,$$

uniformly for $|\delta| \leq \delta_0$. Therefore for any sufficiently small, fixed $c \neq 0$, the function $\gamma(\delta, c)/c$ maps $|\delta| \leq \delta_0$ into itself, and since it is a continuous function of δ it must have at least one fixed point $\delta = \bar{\delta}(c)$, which guarantees branching from λ^0 . Observe of course that the solution could be $\bar{\delta} = 0$, which would be the case for a linear operator A . In any event, we have proved part of the Leray–Schauder theorem (Theorem 2).

To proceed with more concrete information on the branching we shall assume that L is symmetric and that (H₃) holds.

$$(H_3) \quad Rh = Ch + Dh,$$

where

$$C(\tau h) = \tau^p Ch, \quad p \text{ an integer } \geq 2;$$

also there exists a constant K such that, for all h_1, h_2 ,

$$\begin{aligned} \|Ch_1 - Ch_2\| &\leq K \|h_1 - h_2\| (\|h_1\|^{p-1} + \|h_2\|^{p-1}), \\ \|Dh_1 - Dh_2\| &\leq K \|h_1 - h_2\| (\|h_1\|^p + \|h_2\|^p). \end{aligned}$$

Remark 9.1. We are assuming that the leading nonlinear term is a homogeneous polynomial of degree $p \geq 2$. The Lipschitz conditions on C and D are needed for technical reasons.

Since L is symmetric we may choose \mathfrak{N} and $\mathfrak{N} = \mathfrak{N}^\perp$ as our complementary subspaces. Equations (9.10) and (9.12) simplify further to

$$(9.13) \quad w = \delta Tw - TR \left(\sum_{j=1}^k c_j \varphi_j + w \right),$$

$$(9.14) \quad \delta c_i = \left\langle \varphi_i, R \left(\sum_{j=1}^k c_j \varphi_j + w \right) \right\rangle, \quad i = 1, \dots, k.$$

Under (H_3) , we can easily show that the fixed point \bar{w} of (9.13) satisfies

$$\bar{w}(c_1, \dots, c_k, \delta) = o(\|v\|) = o(\varepsilon^p),$$

uniformly in $\delta, |\delta| \leq \delta_0$, where $\varepsilon = \sqrt{c_1^2 + \dots + c_k^2}$. This estimate considerably simplifies the analysis of the branching equation. We consider only the case of a simple eigenvalue, so that

$$\begin{aligned} \delta c &= \langle \varphi, R(c\varphi + \bar{w}) \rangle = \langle \varphi, C(c\varphi + \bar{w}) \rangle + \langle \varphi, D(c\varphi + \bar{w}) \rangle \\ &= \langle \varphi, C(c\varphi) \rangle + \langle \varphi, C(c\varphi + \bar{w}) - C(c\varphi) \rangle \\ &\quad + \langle \varphi, D(c\varphi) \rangle + \langle \varphi, D(c\varphi + \bar{w}) - D(c\varphi) \rangle. \end{aligned}$$

By hypothesis,

$$\begin{aligned} \|C(c\varphi + \bar{w}) - C(c\varphi)\| &\leq K\|\bar{w}\|(\varepsilon^{p-1} + \|\bar{w}\|^{p-1} + \varepsilon^{p-1}) = o(\varepsilon^{2p-1}), \\ \|D(c\varphi)\| &\leq K\varepsilon^{p+1}, \\ \|D(c\varphi + \bar{w}) - D(c\varphi)\| &\leq K\|\bar{w}\|(\varepsilon^p + \|\bar{w}\|^p + \varepsilon^p) = o(\varepsilon^{2p}). \end{aligned}$$

Therefore, by Schwarz's inequality,

$$\delta c = c^p \langle \varphi, C\varphi \rangle + o(\varepsilon^{p+1})$$

or

$$\delta c = c^p [\langle \varphi, C\varphi \rangle + \psi(c, \delta)],$$

where $\psi(c, \delta) \rightarrow 0$ as $c \rightarrow 0$ and $\delta \rightarrow 0$.

If $\langle \varphi, C\varphi \rangle \neq 0$, we can easily show that the solution of

$$(9.15) \quad \delta = c^{p-1} \langle \varphi, C\varphi \rangle$$

gives the leading term in the δ, c relation near the origin. Clearly branching always occurs in this case and the results are displayed in Fig. 12.

Remark 9.2. Branching always occurs for simple eigenvalues, but our method fails if $\langle \varphi, C\varphi \rangle = 0$. One may have to go to higher terms or the branch may not be

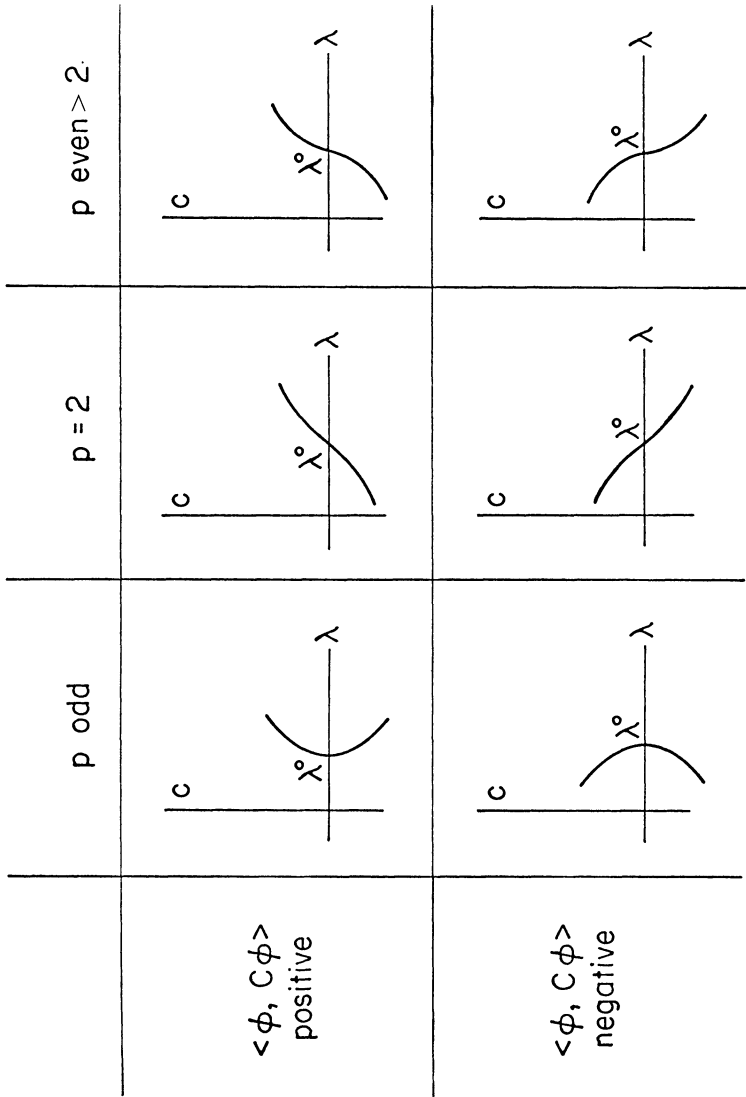


FIG. 12

expressible as a function of c . For instance if A is linear, (9.14) becomes $\delta c = 0$, which has the nontrivial solution $\delta = 0$ representing a branch perpendicular to the λ -axis.

Remark 9.3. If the eigenvalue is not simple, the analysis is more complicated (see, for instance, Sather [94]). Solutions can be constructed by adapting an old device known variously as Newton's polygon or diagram, or as the method of Puiseux (see Hille [39], Pimbley [85], Dieudonné [29]).

10. Example treated by the method of Lyapunov–Schmidt and Poincaré–Keller. Consider the nonlinear problem

$$(10.1) \quad -u'' = \mu \sin u, \quad 0 < x < 1; \quad u'(0) = u(1) = 0.$$

This is essentially the buckling problem (1.6) except for the boundary condition at $x = 1$. Every solution of (1.6) corresponding to $\mu > 0$ and $|\varphi(0)| < \pi$ generates a solution of (10.1) by chopping off the last quarter-wave and rescaling; furthermore every solution of (10.1) can be obtained in this way. Since the linearized problem corresponding to (10.1),

$$(10.2) \quad -u'' = \mu u, \quad 0 < x < 1; \quad u'(0) = u(1) = 0,$$

does not have 0 as an eigenvalue, the translation into an integral equation is technically a little simpler than for (1.6). The eigenvalues of (10.2) are just $\mu_n = (2n - 1)^2 \pi^2 / 4$, $n = 1, 2, \dots$, with corresponding normalized eigenfunctions $\sqrt{2} \cos(2n - 1)\pi x / 2$.

The Green's function $g(x, t)$ that will serve as kernel satisfies

$$-\frac{d^2 g}{dx^2} = \delta(x - t), \quad 0 < x, t < 1; \quad \frac{dg}{dx}(0, t) = g(1, t) = 0,$$

so that

$$g(x, t) = 1 - \max(x, t).$$

The boundary value problem (10.1) is therefore equivalent to the integral equation

$$(10.3) \quad \lambda u(x) = \int_0^1 g(x, t) \sin u(t) dt, \quad \lambda = 1/\mu.$$

By multiplying (10.1) by u' and integrating from 0 to 1, we can show that $\mu \leq 0$ implies $u \equiv 0$. Therefore (10.3) has only the trivial solution for negative λ . Also,

$$|\lambda| \|u\| \leq \|G\| \|\sin u\| \leq \|G\| \|u\|,$$

where $\|G\|$ is the norm of the integral operator generated by the kernel $g(x, t)$. As is well known, $\|G\|$ is the reciprocal of the lowest eigenvalue of the linear problem (10.2); thus $\|G\| = 4/\pi^2$. From the last inequality it follows that nontrivial solutions of (10.3) can occur only for $|\lambda| \leq 4/\pi^2$. This is of course a global result not related to branching theory.

We now apply the Lyapunov–Schmidt method to (10.3). The only possible nonzero branch-points are the simple eigenvalues $\lambda_n = 4/\pi^2(2n - 1)^2$ of the linearized problem

$$\lambda u = \int_0^1 g(x, t)u(t) dt.$$

We write

$$\begin{aligned} Au &= \int_0^1 g(x, t) \sin u(t) dt = \int_0^1 g(x, t)u(t) dt + \int_0^1 g(x, t)(\sin u - u) dt \\ &= Lu + Ru, \end{aligned}$$

and

$$Ru = Cu + Du = - \int_0^1 g(x, t) \frac{u^3(t)}{6} dt + \int_0^1 g(x, t) \left(\sin u - u + \frac{u^3}{6} \right) dt.$$

Now the nonlinear integral operators C and D are not defined on all of $\mathfrak{H} = L_2$; but since it suffices to look for continuous solutions we may confine ourselves to the Banach space of continuous functions. We proceed with the calculation of the eigenvalue branches from (9.15) with $\delta = \lambda - \lambda_n$, $u = c\varphi + w$, where $\|w\|$ is $o(|c|^3)$, $\varphi = \sqrt{2} \cos(2n - 1)\pi x/2$,

$$\delta = -\frac{c^2}{6} \int_0^1 \varphi(x) dx \int_0^1 g(x, t)\varphi^3(t) dt.$$

Since g is symmetric and φ is an eigenfunction corresponding to λ_n , we have $\int_0^1 g(x, t)\varphi(x) dx = \lambda_n\varphi(t)$, so that

$$(10.4) \quad \delta = -\frac{c^2}{6} \lambda_n \int_0^1 \varphi^4(t) dt = -\frac{c^2}{4} \lambda_n = -\frac{c^2}{\pi^2(2n - 1)^2}.$$

The change in μ is easily calculated:

$$(10.5) \quad \mu - \mu_n = \frac{1}{\lambda} - \frac{1}{\lambda_n} = -\frac{\delta}{\lambda\lambda_n} \sim -\frac{\delta}{\lambda_n^2} = \frac{c^2}{16} \pi^2(2n - 1)^2.$$

The branching diagram c versus μ consists of branches, initially parabolic, that emanate from each μ_n and point to the right, very much as the solid curves in Fig. 5.

With the existence theorems of § 9 providing solid foundation, there is no need to treat (10.1) through an integral equation formulation. This saves a certain amount of work, particularly when higher order approximations are desired. Writing $u = c\varphi + w$ in (10.1), we find that w satisfies

$$(10.6) \quad \begin{aligned} -w'' - \mu_n w &= (\mu - \mu_n)(c\varphi + w) + \mu[\sin(c\varphi + w) - (c\varphi + w)], \\ w'(0) &= w(1) = 0. \end{aligned}$$

Consistency requires that the right side of the differential equation be orthogonal to φ ; therefore

$$(10.7) \quad c(\mu - \mu_n) = -\mu \langle \sin(c\varphi + w) - (c\varphi + w), \varphi \rangle,$$

which can be shown to be equivalent to (9.14). To the usual order we find from (10.7),

$$c(\mu - \mu_n) = \frac{\mu c^3}{6} \int_0^1 \varphi^4 dt,$$

or

$$c(\lambda - \lambda_n) = -\lambda_n \frac{c^3}{6} \int_0^1 \varphi^4 dt,$$

which is precisely (10.4).

Another perturbation approach was suggested by Poincaré and fully developed by J. Keller [51]. The method is closely linked to the existence theorem for initial value problems of ordinary differential equations. We illustrate the procedure on (10.1). We expect that solutions of small norm branch from $u \equiv 0$ at $\mu = \mu_n$. We wish to introduce a numerical scale factor that can serve easily as a perturbation parameter. Since any nontrivial solution of (10.1) has $u(0) \neq 0$, we may set $u(x) = az(x)$, where $z(0) = 1$ and the amplitude a will serve as a perturbation parameter that will be small for solutions near the branch-point.

Consider the following initial value problem for a function z :

$$(10.8) \quad z'' + \mu \frac{\sin az}{a} = 0, \quad z(0) = 1, \quad z'(0) = 0,$$

where primes denote differentiation with respect to x .

Problem (10.8) has one and only one solution $z(x, \mu, a)$, depending analytically on μ and a . We define $z(x, \mu, 0)$ by continuity and $z(x, \mu, 0)$ satisfies the linearized problem

$$u'' + \mu u = 0, \quad u(0) = 1, \quad u'(0) = 0,$$

so that

$$(10.9) \quad z(x, \mu, 0) = \cos \sqrt{\mu} x.$$

Every nontrivial solution of (10.1) generates a solution of (10.8) with $a = u(0)$, $z = u(x)/u(0)$. Conversely, any solution of (10.8) with $a \neq 0$, that also satisfies the boundary condition at the right end, yields a nontrivial solution of (10.1) with $u(x) = az$. We therefore look for a relation between μ and a so that the solution of (10.8) also obeys

$$(10.10) \quad z(1, \mu, a) = 0.$$

Of course we have $z(1, \mu_n, 0) = 0$; for branching we wish to find solutions of (10.10) with μ near μ_n and a near 0, $a \neq 0$. The function $z(1, \mu, a)$ is unambiguously determined but not usually explicitly calculable. In our case $z(1, \mu, a)$ can be expressed in terms of elliptic functions, but we do not want to take advantage of this. Instead we want to calculate the solutions (μ, a) of (10.10) near $(\mu_n, 0)$ by applying results on implicit functions.

Setting $z(1, \mu, a) = b(\mu, a)$, our first task is to calculate $b_a(\mu_n, 0)$ and $b_\mu(\mu_n, 0)$, where the subscript denotes differentiation with respect to that variable. It is clear

that $z(x, \mu, a) = z(x, \mu, -a)$ so that $b_a(\mu_n, 0) = 0$. The result can also be obtained by differentiating (10.8) with respect to a . To calculate $b_\mu(\mu_n, 0)$ we first differentiate (10.8) with respect to μ ,

$$z_\mu'' + \frac{\sin az}{a} + \mu(\cos az)z_\mu = 0, \quad z_\mu(0, \mu, a) = 0, \quad z_\mu'(0, \mu, a) = 0,$$

and upon setting $\mu = \mu_n, a = 0$,

$$(10.11) \quad z_\mu'' + \mu_n z_\mu = -z(x, \mu_n, 0) = -\cos \sqrt{\mu_n} x.$$

If $z_\mu(1, \mu_n, 0)$ were 0, it would mean that (10.11) with the boundary conditions $z_\mu'(0, \mu_n, 0) = z_\mu(1, \mu_n, 0) = 0$ would have a solution. Since μ_n is an eigenvalue of the self-adjoint homogeneous problem corresponding to (10.11) and these boundary conditions, the consistency condition

$$\int_0^1 -(\cos \sqrt{\mu_n} x) \varphi(x) dx = 0$$

would have to hold, and this is clearly impossible, since $\varphi(x)$ is proportional to $\cos \sqrt{\mu_n} x$. Therefore $z_\mu(1, \mu_n, 0)$ is not 0. Of course the result could be obtained more simply in this special case by using (10.9) to find

$$z_\mu(1, \mu_n, 0) = -\sqrt{\mu_n} \sin \sqrt{\mu_n} = (-1)^n \frac{(2n-1)\pi}{2}.$$

Thus we know that $b_\mu(\mu_n, 0) \neq 0$. By the implicit function theorem there is a unique function $\mu = \mu(a)$ defined near $a = 0$, such that $\mu(0) = 0$ and $b(\mu(a), a) = 0$. This unambiguous function $\mu(a)$ generates a solution $az(x, \mu(a), a)$ of (10.1) which completely describes the branching at $\mu = \mu_n$. With the suitable expansion parameter a , we are now prepared to find a regular perturbation solution of (10.1) in the form

$$\mu = \mu_n + a^2 \bar{\mu}_n + o(a^2), \quad z = \varphi_n + a^2 \bar{\varphi}_n + o(a^2),$$

where we have used the evenness of μ and z as functions of a . The equation for the leading terms $\bar{\varphi}_n$ and $\bar{\mu}_n$ is

$$\bar{\varphi}_n'' + \mu_n \bar{\varphi}_n = -\mu_n \varphi_n + \mu_n \varphi_n^3/6, \quad \bar{\varphi}_n'(0) = \bar{\varphi}_n(1) = 0.$$

Since μ_n is an eigenvalue of the homogeneous equation, the consistency condition

$$-\bar{\mu}_n \int_0^1 \varphi_n^2 dx + \mu_n \int_0^1 \frac{\varphi_n^4}{6} dx = 0$$

must hold. Therefore,

$$-\frac{1}{2} \bar{\mu}_n + \mu_n/16 = 0,$$

or

$$\bar{\mu}_n = \mu_n/8 = (2n-1)^2 \pi^2/32,$$

and

$$(10.12) \quad \mu - \mu_n \sim a^2 (2n-1)^2 \pi^2/32,$$

which is in agreement with (10.5) if we take into account the different normalizations. Indeed, $\varphi_n = \cos \sqrt{\mu_n}x$, $\varphi(x) = \sqrt{2} \cos \sqrt{\mu_n}x$, and (10.12) yields to first nonvanishing order,

$$u \sim a\varphi_n = \frac{\sqrt{32}}{\pi(2n-1)} \sqrt{\mu - \mu_n} \cos \sqrt{\mu_n}x,$$

whereas (10.5) gives

$$u \sim c\varphi = \frac{4}{\pi(2n-1)} \sqrt{\mu - \mu_n} \sqrt{2} \cos \sqrt{\mu_n}x.$$

The Poincaré–Keller method is very suitable for ordinary differential equations since it uses the amplitude a which is a natural perturbation parameter. The calculations are simpler and higher order terms easier to compute. Unfortunately, the theoretical background relies on existence theory for initial value problems, which limits the scope of rigorous applications.

11. The Taylor problem. A wealth of experimental evidence shows that a steady viscous flow may become unstable as some controllable parameter exceeds a critical value. In some cases the postcritical flow is turbulent and therefore highly unsteady, but in other cases a new steady flow of a cellular type may appear. The best known examples of this latter phenomenon are the Taylor vortices for flow between rotating cylinders and the Bénard cells in fluid layers heated from below.

The usual procedure for finding the critical value of the parameter is to investigate the effect of small disturbances on the original steady flow. The lowest value of the parameter which causes such small disturbances to increase in time is then regarded as the critical value of the parameter. Such a method fails to predict the presence of a new steady flow and cannot calculate the nature of the postcritical flow. We shall investigate another approach due to Velte [109], which at least proves the existence of a second steady flow for the Taylor problem. This, then, is an example of nonuniqueness for the steady Navier–Stokes equations (see also Iudovich [41], Kirchgässner [56], and the review article by Görtler and Velte [34]).

A viscous fluid occupies the space between concentric circular cylinders, the outer one being at rest and the inner one rotating with angular velocity ω . The steady motion of the fluid is governed by the Navier–Stokes equations whose non-dimensional form is

$$(11.1) \quad q \cdot \text{grad } q - \frac{1}{\lambda} \nabla^2 q = -\text{grad } p, \quad \text{div } q = 0,$$

where q is the vector velocity, p the scalar pressure, and λ a physical parameter—the Reynolds number. The boundary condition at a wall states that there is no relative velocity between fluid and wall. The particular geometry clearly calls for cylindrical coordinates r, φ, z with respective velocity components u, v, w . The scaling is such that the inner radius is 1, the outer radius a , and the Reynolds number is proportional to ω . An elementary solution of the nonlinear boundary

value problem is the familiar circular *Couette flow*:

$$(11.2) \quad u_0 = 0, \quad v_0 = \frac{1}{a^2 - 1} \left(\frac{a^2}{r} - r \right), \quad w_0 = 0, \quad p_0 = \int \frac{v_0^2}{r} dr.$$

This flow is a solution for all values of the Reynolds number. The experimental evidence (see Coles [25] for a particularly thorough discussion) shows that for low values of ω Couette flow is stable in the presence of small disturbances and first becomes unstable when ω reaches a critical value ω_1 . The new motion for $\omega > \omega_1$ is itself laminar, consisting of toroidal vortices spaced regularly in the axial direction. As ω increases further a new instability occurs at $\omega = \omega_2$ leading to a flow in which the axially-periodic vortices are simultaneously periodic traveling waves in the azimuth. A general discussion can be found in Davey, DiPrima and Stuart [28]. We shall be interested only in the first critical value of ω (or λ), that is, in branching from Couette flow into a flow independent of φ .

Introducing new variables u, v, w, p to denote the differences between the total flow variables and the Couette flow variables, we find on substitution in (11.1) that u, v, w, p satisfy the equations

$$(11.3) \quad \begin{aligned} uu_r + wu_z - \frac{1}{r}v^2 - \frac{2}{r}vv_0 &= -p_r + \frac{1}{\lambda} \left(\nabla^2 u - \frac{u}{r^2} \right), \\ uv_r + wv_z + \frac{1}{r}uv + \frac{1}{r}(v_0 + rv'_0)u &= \frac{1}{\lambda} \left(\nabla^2 v - \frac{v}{r^2} \right), \\ uw_r + ww_z &= -p_z + \frac{1}{\lambda} \nabla^2 w, \\ (ru)_r + rw_z &= 0, \end{aligned}$$

where the subscript indicates differentiation with respect to that variable. We are now looking for nontrivial solutions (u, v, w, p) of small norm of (11.3), that is, for branching from the basic solution $u = v = w = p = 0$, which corresponds to Couette flow in the new variables. The problem has been treated in a number of ways; here we follow Velte's work where the principal interest is in proving the existence of a branch of the type found experimentally rather than in studying quantitatively the new flow. Introducing the stream function $f(r, z)$ from the divergence equation, we can eliminate the pressure by cross-differentiation to obtain (assuming no net flow through a cross section)

$$(11.4) \quad \begin{aligned} \frac{1}{\lambda} B^2 f + \alpha(r)v_z + M(f, v) &= 0, \\ \frac{1}{\lambda} Bv + \beta f_z + N(f, v) &= 0, \end{aligned}$$

Boundary conditions: $f = f_r = v = 0$ at $r = 1, a$.

The operator B is linear and of the second order,

$$Bf = \nabla^2 f - \frac{f}{r^2} = \frac{1}{r}(rf)_r + f_{zz} - \frac{f}{r^2},$$

and M and N are homogeneous polynomials of degree 2 in v, f and their derivatives of order less than or equal to 3 and 1, respectively:

$$M(f, v) = -(f_z Bf)_r + \left[\frac{1}{r}(rf)_r Bf \right]_z + \frac{1}{r^2}(v^2)_z,$$

$$N(f, v) = -\frac{1}{r}(rv)_r f_z + \frac{1}{r}v_z(rf)_r$$

and

$$\alpha(r) = \frac{2v_0(r)}{r}, \quad \beta = \frac{2}{a^2 - 1}.$$

Obviously $f = v = 0$ is a solution of (11.4) for all λ . We shall look for branching into a solution with period l in the z -direction. A naive approach is to linearize (11.4) about $f = v = 0$ by striking out the nonlinear terms M and N . The resulting linear equations may have a smallest eigenvalue λ_1 which can be expected to be a branch-point of the nonlinear problem. This is all very nice but unfortunately is not backed by theorems. To proceed rigorously we shall reformulate the problem (11.4) in terms of completely continuous operators to enable us to apply the Leray–Schauder theory (Theorem 2). Let $-L_1$ be the inverse of the fourth order operator B^2 with the boundary conditions $f = f_r = 0$ at $r = 1, a$; let $-L_2$ be the inverse of B with the boundary conditions $v = 0$ at $r = 1, a$. There are many choices of the underlying, z -periodic, function spaces that turn these inverses into compact operators. The particular choice of these spaces will be dictated by other considerations, but, in any event, (11.4) takes the form

$$(11.5) \quad \begin{aligned} \lambda L_1[\alpha v_z + M(f, v)] - f &= 0, \\ \lambda L_2[\beta f_z + N(f, v)] - v &= 0. \end{aligned}$$

If the function spaces are correctly chosen, the Fréchet derivatives of the quadratic operators M and N are in fact 0 at $f = v = 0$. This actually requires a minor modification of the norm proposed by Velte, but the rest of his development needs no change. The linearized problem corresponding to (11.5) is then

$$(11.6) \quad \lambda L_1(\alpha v_z) = f, \quad \lambda L_2(\beta f_z) = v.$$

Branching from $f = v = 0$ will be established if (11.6) has a *simple* eigenvalue. In view of the symmetry of the equations we can restrict ourselves to stream functions that are odd in z and circumferential velocities that are even in z . With $\sigma = 2\pi/l$, we develop f and v in the trigonometric series

$$f(r, z) = \sum_{n=1}^{\infty} f_n(r) \sin n\sigma z,$$

$$v(r, z) = \sum_{n=0}^{\infty} v_n(r) \cos n\sigma z.$$

Substitution in (11.6) is equivalent to substitution in (11.4) with omission of nonlinear terms. This yields the infinite system of ordinary linear differential equations

$$(11.7) \quad \begin{aligned} [S - (n\sigma)^2]f_n(r) &= \lambda\alpha(r)n\sigma v_n(r), \\ -[S - (n\sigma)^2]v_n(r) &= \lambda\beta n\sigma f_n(r), \end{aligned}$$

where

$$S = \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right) - \frac{1}{r^2},$$

and the boundary conditions require f_n , f'_n , and v_n to vanish at $r = 1, a$.

Let us look for the moment at the pair of equations

$$(11.8) \quad \begin{aligned} (S - k^2)f &= \lambda\alpha(r)kv(r), \\ -(S - k^2)v &= \lambda\beta kf(r), \end{aligned}$$

with f , f' , v vanishing at $r = 1, a$.

The system (11.8) is just the one that arises in linear stability theory. By using Green's functions one can translate (11.8) first into a pair of integral equations and then, by eliminating v , into a single integral equation having the form

$$(11.9) \quad f(r) = \mu \int_1^a h(r, s; k) f(s) ds, \quad \mu = \lambda^2.$$

The kernel h is positive in the open square $1 < r, s < a$. From Jentzsch's theorem, it follows that (11.9) has an eigenfunction $f_1(r)$ that is positive in $1 < r < a$. The corresponding eigenvalue μ_1 is real and positive and any other eigenfunction belongs to an eigenvalue whose absolute value exceeds μ_1 . Thus with $\lambda_1 = \sqrt{\mu_1}$ and v_1 expressed in terms of f_1 from the integral equation corresponding to (11.8b), we have a solution of (11.8). Setting $k = \sigma$, we also have a solution λ_1 , f_1 , v_1 of (11.7) (the other v_n , f_n being set equal to 0). By a clever argument, Velte is able to show that σ can be chosen so that this eigenvalue λ_1 cannot belong to any other of the equations (11.7) with $n = 2, 3, \dots$. Thus for an appropriate choice of σ , (11.7) has a single eigenfunction corresponding to its lowest positive eigenvalue. This in turn means that in the space of $2\pi/\sigma$ -periodic functions, (11.6) has a lowest positive eigenvalue λ_1 to which corresponds a single eigenfunction. This does not guarantee that λ_1 is simple in the sense required by Theorem 2. Our operators are not symmetric so that we must still prove that the Riesz index is unity. The proof is given in Velte and is omitted here. By using a Lyapunov-Schmidt procedure, Kirchgässner [56] has obtained more detailed information on the solution. He has, for instance, shown that the branching is to the right and that it differs asymptotically from the Couette flow by terms having a λ -dependence $\sqrt{\lambda - \lambda_1}$. In a recent paper, Kirchgässner and Sorger [57] have analyzed the stability of the new branch by using results of Prodi [87].

Similar methods have also been used to study the onset of convection (see Velte [108], Ukhovskii and Iudovich [104], Iudovich [42], Fife and Joseph [31], Fife [30], Rabinowitz [89], Joseph [43], [44], Segel [98], Busse [21]).

12. Buckling of a thin rod. We shall investigate the buckling of a pin-ended, thin rod (see Fig. 1) under more general constitutive laws than in § 1. J. M. Greenberg [36] considered this problem and showed that there exist reasonable materials for which the rod has only a finite number of critical buckling loads. Antman [4] noted a similar phenomenon for curved rods. In Greenberg's treatment the constitutive laws are not in accord with the known forms of the equilibrium equations for a plane elastica (see Antman and Warner [8], A. E. Green [35]). Greenberg's analysis is readily modified to take account of compatible constitutive laws.

A material point whose initial coordinates were $x = X$, $y = 0$ is found after buckling at $x = X + u(X)$, $y = v(X)$, where u and v are as smooth as necessary and satisfy the boundary conditions $u(0) = v(0) = v(l) = 0$. The functions u and v completely describe the deformation, but it will be more convenient to use the pair of functions v and φ , where φ is the angle in Fig. 2. As strain measures we choose

$$\delta = ds/dX \quad \text{for elongation,}$$

$$\mu = -d\varphi/dX \quad \text{for bending.}$$

The more usual measure of bending is the curvature $\kappa = -d\varphi/ds = \mu/\delta$. The reason for using μ instead of κ has to do with the equilibrium equations of a plane elastica which have the form

$$(12.1) \quad M = \partial W / \partial \mu, \quad N = \partial W / \partial \delta,$$

where $W(\mu, \delta)$ is a strain energy function, M the bending moment and N the axial force (see Fig. 2). Other considerations aside, (12.1) is compatible with the pair of constitutive equations $M = \hat{M}(\mu)$ and $N = \hat{N}(\delta)$, where \hat{M} and \hat{N} are arbitrary. On the other hand a pair of the type $M = \hat{M}(\kappa)$, $N = \hat{N}(\delta)$, as considered by Greenberg, would violate (12.1). We shall therefore adopt a pair,

$$(12.2) \quad M = \hat{M}(\mu), \quad N = \hat{N}(\delta),$$

that we assume can be solved for μ and δ to give

$$(12.3) \quad \mu = \hat{\mu}(M), \quad \delta = \hat{\delta}(N),$$

where $\hat{\mu}$ is an odd function with positive derivative and $\hat{\delta}$ is positive and monotonically decreasing with $\hat{\delta}(0) = 1$ (see Fig. 13). The case of the Euler-Bernoulli, inextensible rod of § 1 corresponds to $\hat{\delta} \equiv 1$, $\hat{\mu}(M) = M/(EI)$, since now $s = X$.

In addition to the constitutive equations (12.3), we have the geometric relation

$$(12.4) \quad dv/ds = \sin \varphi$$

and the equilibrium equations

$$(12.5) \quad N = P \cos \varphi$$

and

$$(12.6) \quad M = Pv.$$

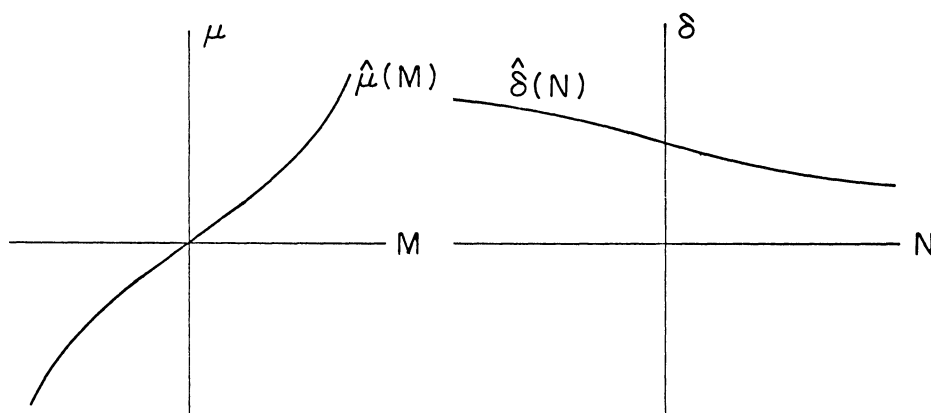


FIG. 13

From (12.6) we deduce $\mu = \hat{\mu}(Pv)$, and from (12.5), $\delta = \hat{\delta}(P \cos \varphi)$. Using (12.4) and the definitions of δ and μ , we obtain the pair of nonlinear differential equations of the first order in v and φ :

$$(12.7) \quad -\frac{d\varphi}{dX} = \hat{\mu}(Pv), \quad \frac{dv}{dX} = \hat{\delta}(P \cos \varphi) \sin \varphi,$$

with the boundary conditions $v(0) = v(l) = 0$. For any value of P , a particular solution of (12.7) is $v \equiv 0$, $\varphi \equiv 0$, which corresponds to pure compression with $\delta = \text{const.} = \hat{\delta}(P)$. Although principally interested in the branching from this compressive solution, we shall carry out much of the analysis globally and explicitly.

The equations (12.7) have the same symmetry properties as the example in § 1. It will therefore suffice to consider the initial value problem for (12.7) with $P > 0$, $v(0) = 0$, $\varphi(0) = \alpha$, $0 < \alpha < \pi$, and then to choose those solutions that also satisfy the boundary condition $v(l) = 0$. Under rather mild assumptions on $\hat{\mu}$ and $\hat{\delta}$, the initial value problem has one and only one solution (v, φ) which oscillates about $(0, 0)$. The graph of v and φ over a period consists of 4 congruent half-bays resembling a sine wave and a cosine wave, respectively. Quantitative information can be found by multiplying the first of (12.7) by dv/dX , the second by $d\varphi/dX$, and adding. After integration, we obtain

$$(12.8) \quad A(Pv) = B(P \cos \varphi) - B(P \cos \alpha),$$

where

$$A(t) = \int_0^t \hat{\mu}(z) dz, \quad B(t) = \int_0^t \hat{\delta}(z) dz.$$

For small t , $A(t) \sim \hat{\mu}'(0)t^2/2$ and $B(t) \sim t$. Since B is monotonically increasing and $A \geq 0$, we have $\cos \varphi \geq \cos \alpha$ or $|\varphi| \leq \alpha$. The half-period T of oscillations

is found from

$$\begin{aligned}
 (12.9) \quad T &= -2 \int_0^\alpha \frac{dX}{d\varphi} d\varphi = 2 \int_0^\alpha \frac{1}{\hat{\mu}(Py)} d\varphi \\
 &= 2 \int_0^\alpha \frac{d\varphi}{\hat{\mu}\{A^{-1}[B(P \cos \varphi) - B(P \cos \alpha)]\}}.
 \end{aligned}$$

The singularity at $\varphi = \alpha$ is integrable, and the limiting half-period of small oscillations ($\alpha \rightarrow 0+$) can be calculated directly from (12.9) or from the linearized problem

$$-\frac{d\varphi}{dX} = \hat{\mu}'(0)Pv, \quad \frac{dv}{dX} = \varphi\hat{\delta}(P)$$

whose solution has the half-period $\pi[\hat{\mu}'(0)P\hat{\delta}(P)]^{-1/2}$.

Branching from the compressive solution of (12.7) can occur if and only if l is an integral multiple of a half-period, that is,

$$(12.10) \quad \hat{\mu}'(0)P\hat{\delta}(P) = n^2\pi^2/l^2, \quad n = 1, 2, 3, \dots$$

Equation (12.10) determines the critical loads. From Fig. 13 it is clear that there are many possible behaviors for $P\hat{\delta}(P)$. If the rod is inextensible or sufficiently resistant to compression, $P\hat{\delta}(P) \rightarrow \infty$ as $P \rightarrow \infty$. This means that (12.10) has an infinite number of roots P_n , that is, an infinite number of critical loads. If the material is soft enough in compression, we could have $P\hat{\delta}(P) \rightarrow 0$ as $P \rightarrow \infty$, which yields at most a finite number of buckling loads. In fact if the maximum of $\hat{\mu}'(0)P\hat{\delta}(P)$ is less than π^2/l^2 , no buckling occurs regardless of the compressive load. This amusing result may or may not be physically significant. High compression of a soft material will make it bulge, thereby ruining the assumption that the rod is thin and of homogeneous cross section. Such bulging is perhaps interpretable as a form of skin buckling that occurs in 3-dimensional problems. The reader is also referred to the papers by Antman [6], [7] and by Tadjbakhsh and Odeh [102].

13. Chain reaction in a rod. We shall consider a one-dimensional version of a more complicated neutron transport problem investigated by Pazy and Rabino-witz [80]. A thin rod, $0 < x < a$, is regarded as a simple nuclear reactor. Between collisions with atoms of the reactor, neutrons are assumed to move with unit velocity along the rod. Neutrons do not interact with one another. When a neutron collides with a fixed atom, the outcome is instantaneous replacement of the original neutron by 0, 1, \dots , or N neutrons, the respective probabilities being denoted by c_0, c_1, \dots, c_N , with

$$(13.1) \quad \sum_{k=0}^N c_k = 1, \quad c_k \geq 0.$$

The neutrons arising from a collision are equally likely to be moving to the left or the right with unit velocity. The expected number of neutrons arising from a collision is the *multiplying factor*

$$(13.2) \quad c = \sum_{k=1}^N kc_k.$$

Collisions of neutrons with atoms are assumed to be governed by the Poisson process. In an infinitesimal interval $(x, x + \Delta)$, there is a probability $\sigma\Delta$ of a collision with an atom, where σ is a given positive physical constant—the reciprocal of the mean free path. The probability of no collision in a path of length b is therefore $e^{-\sigma b}$. When a neutron reaches the end of the rod, it is absorbed.

A single neutron moving to the right is introduced at the point x at $t = 0$. We want to calculate the probability $u(x)$ that at least 1 neutron is alive at $t = \infty$. The quantity $v(x)$ is similarly defined for a neutron initially moving to the left. Clearly, $v(x) = u(a - x)$. On physical grounds one can infer that if σ is large (small mean free path) there will be nearly certain survival if $c > 1$, nearly certain extinction if $c < 1$. If σ is small, the neutron will nearly always be absorbed at the ends of the rod unless the multiplying factor c is very large. We now derive a governing equation for $u(x)$, or rather, for the extinction probability $1 - u(x)$. There are two roads to extinction: (i) the neutron reaches the end $x = a$ without collision; (ii) the neutron has its first collision in some interval $(y, y + dy)$ and all products ultimately die. Thus

$$1 - u(x) = e^{-(a-x)\sigma} + \int_x^a e^{-(y-x)\sigma} \sigma p(y) dy,$$

where $p(y)$ is the probability of all products of the collision from $(y, y + dy)$ becoming extinct. There is a probability c_k of k neutrons being generated so that

$$p(y) = \sum_{k=0}^N c_k p_k(y),$$

where $p_k(y)$ is the probability that none of the k neutrons generated yield a chain reaction. Of the k neutrons produced there is a probability $\binom{k}{p} \left(\frac{1}{2}\right)^k$ that p will move to the right and the rest to the left; the corresponding extinction probability is $[1 - u(y)]^p [1 - v(y)]^{k-p}$. Therefore,

$$1 - u(x) = e^{-(a-x)\sigma} + \int_x^a \sigma e^{-(y-x)\sigma} \sum_{k=0}^N c_k \sum_{p=0}^k \binom{k}{p} \left(\frac{1}{2}\right)^k [1 - u(y)]^p [1 - v(y)]^{k-p} dy.$$

A similar equation holds for $v(x)$. Setting

$$(13.3) \quad z(x) = (u(x) + v(x))/2,$$

we obtain after some simplification,

$$(13.4) \quad z(x) = \int_0^a E(x, y) G[z(y)] dy = Nz, \quad 0 \leq x \leq a,$$

where

$$(13.5) \quad E(x, y) = (\sigma/2)e^{-\sigma|x-y|},$$

$$(13.6) \quad G(z) = cz - \sum_{k=2}^N c_k[(1-z)^k - 1 + kz] = 1 - \sum_{k=0}^N c_k(1-z)^k.$$

The following properties of the kernel $E(x, y)$ and of the polynomial $G(z)$ play a role in the sequel:

$$(13.7) \quad E(x, y) \geq d > 0 \quad \text{on } 0 \leq x, y \leq a,$$

$$(13.8) \quad \int_0^a E(x, y) dy \leq 1, \quad 0 \leq x \leq a,$$

$$(13.9) \quad G(z) \text{ strictly increasing } 0 \leq z \leq 1, \quad G(0) = 0, \quad G(1) = 1 - c_0 \leq 1.$$

$$(13.10) \quad \begin{aligned} G'(0) = c, \quad G'(1) = c_1 < c, \quad G'(z) \text{ strictly decreasing,} \\ G(z) \text{ strictly concave, } c_2, \dots, c_N \text{ not all zero.} \end{aligned}$$

Equation (13.4) is a nonlinear integral equation of the Hammerstein type. Clearly $z \equiv 0$ is always a solution. In view of (13.3) and the probability definitions of u and v , we shall be interested in nontrivial $z(x)$ satisfying $0 \leq z(x) \leq 1$. We pause to observe that (13.4) is equivalent to the nonlinear differential equation

$$(13.11) \quad \begin{aligned} -\frac{d^2 z}{dx^2} + \sigma^2 z &= \sigma^2 G[z(x)], \quad 0 < x < a, \\ z'(0) - \sigma z(0) &= 0, \quad z'(a) + \sigma z(a) = 0. \end{aligned}$$

If c is regarded as a parameter, (13.4) is nearly the kind of nonlinear problem studied in earlier sections. The linearization of (13.4) about $z = 0$ yields the linear integral equation

$$(13.12) \quad h(x) = c \int_0^a E(x, y)h(y) dy = cLh,$$

where c may be viewed as an eigenvalue parameter. The corresponding linearization of (13.11) is $-h'' + \sigma^2 h = c\sigma^2 h$, with the same boundary conditions. One can obtain branching information for solutions of small norm of (13.4) by using our previous methods, but as has already been stated, we are looking only for positive solutions whose maximum does not exceed 1. We shall use entirely different methods to prove the existence and uniqueness of these solutions. It turns out that the branch that emanates from the lowest eigenvalue of (13.12) corresponds to the desired solution.

Following Pazy and Rabinowitz, we set the problem of solving (13.4) in the Banach space \mathfrak{B} of continuous functions on $0 \leq x \leq a$. Let \mathfrak{S} be the subset of \mathfrak{B} consisting of functions $z(x)$ satisfying $0 \leq z(x) \leq 1$. The nonlinear operator N maps \mathfrak{S} into \mathfrak{S} and can be shown to be compact and continuous on \mathfrak{S} ; moreover, N is monotonic in the sense that if $z, w \in \mathfrak{S}$ and $z \leq w$ on $0 \leq x \leq a$, then $Nz \leq Nw$ on $0 \leq x \leq a$. We seek nontrivial solutions of (13.4) in \mathfrak{S} . A function $\bar{z}(x) \in \mathfrak{S}$ is called a *maximal solution* if $\bar{z} \geq z$ for all z in \mathfrak{S} satisfying (13.4).

THEOREM 7. For all $c > 0$, (13.4) has a maximal solution.

Proof. We construct the solution by successive approximations. Let $z_0(x) = 1$ and let $z_{i+1} = Nz_i$. We have $z_1(x) \leq z_0(x)$, and, successively, $z_2 \leq z_1$, etc. Therefore,

$$0 \leq z_{i+1} = Nz_i \leq Nz_{i-1} = z_i \leq 1.$$

The sequence $z_i(x)$ is a uniformly bounded monotonically decreasing sequence which is easily shown to be equicontinuous. There exists therefore a subsequence converging uniformly to \bar{z} in \mathfrak{S} , and by monotonicity, the original sequence has the same property, so that $N\bar{z} = \bar{z}$ and \bar{z} satisfies (13.4). It is also easy to see that \bar{z} is maximal.

Of course, it might be possible at this stage that $\bar{z} \equiv 0$. To show otherwise, we shall need information about the linearized problem (13.12). The kernel in (13.12) is positive; we can therefore use once more the theorem of Jentzsch (see § 11). Rephrased in the new notation it states that there exists a positive eigenvalue c^* which has smaller modulus than any other eigenvalue and that the corresponding eigenfunction can be chosen positive. In our particular case many of the difficulties of Pazy and Rabinowitz's paper disappear. Our kernel is symmetric and strictly positive; moreover, (13.12) corresponds to a linear self-adjoint, Sturm–Liouville problem. We know therefore that c^* is a simple eigenvalue. We normalize the corresponding eigenfunction h^* by

$$\max_{0 \leq x \leq a} h^*(x) = 1.$$

We note from (13.12) that $c^* \geq 1$. From the linearized version of (13.11) we deduce further that $c^* \rightarrow 1$ as $\sigma \rightarrow \infty$ and $c^* \rightarrow \infty$ as $\sigma \rightarrow 0$, just as conjectured in the early part of the section. Our experience with branching problems guarantees that a branch for (13.4) emanates from $c = c^*$. We show that this branch exists for all $c > c^*$ and belongs to \mathfrak{S} .

THEOREM 8. If $c > c^*$, (13.4) has nontrivial solutions.

Proof. We know that if c exceeds c^* by a small amount, the branch is essentially proportional to h^* . This suggests using an iteration process beginning with a small multiple of h^* to build up to the desired nontrivial solution.

With c fixed, $c > c^*$, we can choose δ , $0 < \delta < 1$, such that

$$(1 - \delta)c/c^* \geq 1.$$

Now $G'(z)$ is continuous and takes on the value c at $z = 0$. With δ already fixed, we may select ε_0 , $0 < \varepsilon_0 \leq 1$, such that

$$G'(z) \geq (1 - \delta)c, \quad 0 \leq z \leq \varepsilon_0.$$

With $0 < \varepsilon < \varepsilon_0$, and recalling that $G(0) = 0$, we find by the mean value theorem,

$$\begin{aligned} N(\varepsilon h^*) &= \int_0^a E(x, y) G[\varepsilon h^*(y)] dy \\ &= \int_0^a E(x, y) G'[w(y)] \varepsilon h^*(y) dy \quad (0 \leq w \leq \varepsilon) \\ &\geq \frac{(1 - \delta)c\varepsilon}{c^*} h^*(x) \geq \varepsilon h^*(x). \end{aligned}$$

Now let $t_0(x) = \varepsilon h^*(x)$ and $t_{i+1}(x) = Nt_i$. Arguments similar to those used in Theorem 7 show that t_i is a monotonically increasing sequence with limit $z(x)$ satisfying $0 \leq \varepsilon h^* \leq z \leq 1$.

This solution z can be shown to be independent of ε and is clearly nontrivial. Since $\bar{z} \geq z$, \bar{z} is itself nontrivial for $c > c^*$. The notation suggests that z is a minimal solution; that is, if z is any solution of (13.4) belonging to \mathfrak{S} , then $z \leq \bar{z}$. Indeed, let z be a nontrivial solution of (13.4) in \mathfrak{S} ; using (13.7),

$$z \geq d \int_0^a G[z(y)] dy > 0,$$

since $G > 0$ for $0 < z \leq 1$. Because h^* is also positive in $0 \leq x \leq a$, we can find $\varepsilon > 0$ such that $0 < \varepsilon h^* \leq z$. The construction used in Theorem 8 will then show that $0 < \bar{z} \leq z$.

Next we would like to show that the only solution in \mathfrak{S} for $c < c^*$ is $z \equiv 0$ and that there is a unique nontrivial solution in \mathfrak{S} for $c > c^*$ (the solution being $z = \bar{z} = z$). The first result depends on the fact that G is concave. Using the mean value theorem and the concavity of G ,

$$\begin{aligned} z &= \int_0^a E(x, y) G[z(y)] dy = \int_0^a E(x, y) G'[w(y)] z(y) dy \\ &\leq \int_0^a E(x, y) G'(0) z(y) dy = cLz. \end{aligned}$$

By taking L_2 -norms or by quoting a theorem of Krein and Rutman, we see that nontrivial solutions are possible only for $c \geq c^*$. Finally we show that $\bar{z} = z$ for $c > c^*$. Since \bar{z} and z both satisfy (13.4), we find on using the symmetry of $E(x, y)$,

$$0 = \int_0^a \{ \bar{z}(x) G[z(x)] - z(x) G[\bar{z}(x)] \} dx,$$

or

$$0 = \int_0^a \bar{z}(x) z(x) \left\{ \frac{G[z(x)]}{z(x)} - \frac{G[\bar{z}(x)]}{\bar{z}(x)} \right\} dx.$$

Of course, we have $z \leq \bar{z}$; if the inequality held for some x interval, then the strict concavity of G would imply that the term in brackets is positive on that interval; but $\bar{z}z > 0$ in $0 < x < a$ so that the integral from 0 to a would be positive. Therefore $z = \bar{z}$ and (13.4) has a unique nontrivial solution in \mathfrak{S} for $c > c^*$.

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