Normal Form for the Fold Bifurcation

The purpose of these notes is to give a simple proof (simpler than Kuznetsov) that the normal form for the fold bifurcation is

$$\frac{dx}{dt} = a \pm x^2 \tag{1}$$

Theorem: If the function $f(y, a)$ is smooth and

1. $f(0, 0) = 0$
2. $f_y(0, 0) = 0$
3. $f_{yy}(0, 0) \neq 0$, $f_a(0, 0) \neq 0$,

then the flow $\frac{dy}{dt} = f(y, a)$ is topologically equivalent to the flow (1) in some sufficiently small neighborhood of the origin $y = 0, a = 0$.

The proof is in three steps. **Step 1:** For $f(y, a)$ there is a function $y = Y(a)$ so that $f_y(Y(a), a) = 0$. This follows immediately from the Implicit Function Theorem. (Proof below)

Let $z = y - Y(a)$. Then

$$\frac{dz}{dt} = g(z, a) = g_0(a) + g_2(a)z^2 + g_3(a)z^3 + \cdots, \tag{2}$$

a power series in z, with coefficients which are functions of a, $g_0(a) = f(Y(a), a)$. Introduce the (invertible) change of parameter $b = g_0(a)$ so that

$$\frac{dz}{dt} = G(z, b) = b + G_2(b)z^2 + G_3(b)z^3 + \cdots. \tag{3}$$

Proof: To find solutions of $f_y(Y(a), a) = 0$, notice that $Y(0) = 0$, since $f_y(0, 0) = 0$. Differentiate with respect to a and find

$$f_{yy}(Y(a), a)Y'(a) + f_{ya}(Y(a), a) = 0 \tag{4}$$

so that

$$f_{yy}(0, 0)Y'(0) + f_{ya}(0, 0) = 0, \tag{5}$$

which means that $Y'(0)$ (hence $Y(a)$) exists and is unique provided $f_{yy}(0, 0) \neq 0$, which it is.

Step 2: There is a linear change of variables $z = \alpha(b) + \beta(b)x$ with $\alpha = O(b), \beta = O(1)$, so that for $F(x, b) = G(\alpha(b) + \beta(b)x, b)$

$$F(r, b) = (G(\alpha(b) + \beta(b)r, b) = 0 \tag{6}$$

on $b = \sigma r^2$, for all sufficiently small r.

The proof follows from the Implicit Function Theorem. In fact, the transformation is (use Maple; see the Maple code at the bottom of these notes)

$$z = \frac{G_{30}}{2f_{20}^4} + \frac{x}{f_{20}}, \tag{7}$$

where

$$G_{20} = \sigma f_{20}^2, \qquad \sigma^2 = 1, \tag{8}$$

to leading order in b.

Furthermore,

$$f(x,b) = (b + \sigma x^2)(1 + \frac{G_{30}x}{\sigma f_{20}^3} + \cdots). \tag{9}$$

Step 3: Suppose $f(\pm\sqrt{-b\sigma}, b) = 0$ for all b with $\sigma b < 0$, where $\sigma = \text{sgn} f_{xx}(0,0)$. Then the function

$$\mu(y,b) = \frac{b + \sigma x^2}{f(x,b)} \tag{10}$$

is a bounded positive function. It follows that the flow $\frac{dx}{dt} = f(x,b)$ is topologically equivalent to the flow $\frac{dx}{dt} = b + \sigma x^2$ since

$$b + \sigma x^2 = \mu(x,b) f(x,b). \tag{11}$$

This same technique can be used to prove the following:

Theorem: Suppose $f(y,a) = ay + y^2 g(y,a)$ and $g(0,0) \neq 0$. Then the normal form for the flow

$$\frac{dy}{dt} = f(y,a) \tag{12}$$

is

$$\frac{dx}{dt} = ax \pm x^2. \tag{13}$$

Theorem: Suppose $f(y,a) = ay + y^3 g(y,a)$ and $g(0,0) \neq 0$. Then the normal form for the flow

$$\frac{dy}{dt} = f(y,a) \tag{14}$$

is

$$\frac{dx}{dt} = ax \pm x^3. \tag{15}$$

Here is the Maple code to do Step 2:

```
# Suppose g(y,b)  = b +G_2(b) *z^2 +  G_3(z,b)*z^3;  The following maple
# code determines the transformation z = c(b) + d(b) x so that
#f(x,b) = g(c(b) + d(b) x,b) has roots at x = +/- sqrt(-b sgn G_2(0))
restart;
# define f;
n:=4;
g:=b +add(add(G||j||k*b^k*z^j,j=2..n),k=0..n);


 z:= add(c||j*b^j,j=1..n)+add(d||j*b^j,j=0..n)*x;

eq1:=subs(b=-s*r^2,x=r,g):
coeff(eq1,r,0);
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coeff(eq1,r,1);

coeff(eq1,r,2);

 d0:=1/f20;

c1:=solve(coeff(eq1,r,3),c1);
d1:= solve(coeff(eq1,r,4),d1);

c2:=solve(coeff(eq1,r,5),c2);

d2:=solve(coeff(eq1,r,6),d2);
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