Notes on resultant and bifurcation analysis

1 The Resultant

Given two polynomials,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, \qquad g(x) = b_m x^m + a_{m-1}x^{m-1} + \cdots + b_0, \tag{1}$$

define the Sylvester matrix as the $(n+m) \times (n+m)$ matrix

$$S = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & 0 & 0 & 0 \\ 0 & a_n & a_{n-1} & \cdots & & & \\ 0 & 0 & a_n & \cdots & a_1 & a_0 & 0 \\ & & & & & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & 0 & & \\ 0 & b_m & b_{m-1} & \cdots & & & \\ 0 & 0 & & & b_1 & b_0 & 0 \\ & & & & & b_1 & b_0 \end{pmatrix} \tag{2}$$

Specific example, let $f = a_1 x + a_0$, $g = b_1 x + b_0$, then

$$S = \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}. \tag{3}$$

Define the resultant of f and g as

$$R(f,g) = det(S). \tag{4}$$

The main theorem, which we are about to prove is
Theorem Suppose $f(x) = a_n \prod_{i=1}^{n}(x - \xi_i)$ and $g(x) = b_m \prod_{i=1}^{m}(x - \eta_j)$. Then

$$\begin{aligned} R(f,g) &= a_n^m b_m^n \prod_i \prod_j (\xi_i - \eta_i) && (5) \\ &= a_n^m \prod_i g(\xi_i) && (6) \\ &= (-1)^{nm} b_m^n \prod_j f(\eta_j). && (7) \end{aligned}$$

An example:

$$f = x^2 + ax + b, \qquad g = f' = 2x + a, \tag{8}$$

then

$$S = \begin{pmatrix} 1 & a & b \\ 2 & a & 0 \\ 0 & 2 & a \end{pmatrix}. \tag{9}$$

and

$$det(S) = 4b - a^2 = 4f(-\frac{a}{2}). \tag{10}$$

Note that

$$\xi_i = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b}), \qquad \eta = -\frac{a}{2}, \tag{11}$$

so that

$$\prod_{i,j}(\xi_i - \eta_j) = -\frac{1}{4}(a^2 - 4b), \tag{12}$$

which agrees with the theorem.

Some interesting formulas:

1. Suppose $\deg(g) = k < m$. Then,

$$R_{n,m}(f,g) = a_n^{m-k} R_{n,k}(f,g) \tag{13}$$

   This follows since $g = 0x^m + 0x^{m-1} + \cdots + b_k x^k + \cdots$.

2. Suppose $\deg(f) = n$, $\deg(g) = m$, $\deg(h) = k$.

   (a) If $m + k \leq n$, then

$$R(f + gh, g) = R(f,g). \tag{14}$$

   (b) If $n + k \leq m$, then

$$R(f, g + fh) = R(f,g). \tag{15}$$

   This follows directly from properties of determinants.

Proof of the main Theorem

We use induction on $m + n$.

First, for $n = m = 1$, $f = a_1 x + a_0$, $g = b_1 x + b_0$,

$$S = \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}, \tag{16}$$

and

$$R(f,g) = a_1 b_0 - b_1 a_0 = a_1 b_1 (-\frac{a_0}{a_1} + \frac{b_0}{b_1}), \tag{17}$$

as required. The case $m = 0$, $n = 1$ works as well by setting $b_1 = 0$ in which case $R(f,g) = a_1 g(-\frac{a_0}{a_1})$ since $g = b_0$.

Now, for some n and m, suppose, without loss of generality, that $m \geq n$. Then, there is a polynomial h with $\deg(h) = m - n$ for which

$$g = hf + r, \tag{18}$$

and $\deg(r) = k < m$. Then,

$$\begin{aligned} R(f,g) = R(f,g-hf) = R_{n,m}(f,r) &= a_n^{m-k} R_{n,k}(f,r) && (19)\\ &= a_n^{m-k} a_n^k \prod_i r(\xi_i) && (20)\\ &= a_n^m \prod_i \Big(g(\xi_i) - h(\xi_i) f(\xi_i)\Big) && (21)\\ &= a_n^m \prod_i g(\xi_i), && (22) \end{aligned}$$

as required. QED

1.1 The Discriminant

For any polynomial f with roots ξ_i, the discriminant is defined as

$$\Delta(f) = \prod_{i<j} (\xi_i - \xi_j)^2. \qquad (23)$$

A calculation: Since $f = a_n \prod_{j=1}^n (x - \xi_j)$, then $f'(\xi_i) = a_n \prod_{j \neq i}^n (\xi_i - \xi_j)$. It follows that

$$\begin{aligned} R(f,f') &= a_n^{n-1} \prod_{i=1}^n f'(\xi_i) && (24)\\ &= a_n^{n-1} a_n^n \prod_{i=1}^n \prod_{j \neq i}^n (\xi_i - \xi_j) && (25)\\ &= a_n^{2n-1} \prod_{i<j} (x_i - x_j)(x_j - x_i) && (26)\\ &= (-1)^{\frac{n(n-1)}{2}} a_n^{2n-1} \prod_{i<j} (x_i - x_j)^2 = (-1)^{\frac{n(n-1)}{2}} a_n^{2n-1} \Delta(f). && (27) \end{aligned}$$

Examples:

1. For the quadratic polynominal $f = x^2 + ax + b$, $f' = 2x + a$,

   $$R(f,f') = 4b - a^2 = -\Delta(f). \qquad (28)$$

   Obviously, if $4b = a^2$, $f = x^2 + ax + \frac{a^2}{4} = (x + \frac{a}{2})^2$ has a double root.

2. For the cubic polynominal $f = x^3 + ax + b$ (note that the quadratic term can always be eliminated with a shift of the variable x), then

   $$R(f,f') = 4a^3 + 27b^2. \qquad (29)$$

   The curve along which $R(f,f') = 0$ is the curve in parameter space along which f has a double zero. This is known as the cusp bifurcation.

2 Examples

We want to use the resultant to understand features of solutions of interesting problems.

2.1 Spruce budworm dynamics

Spruce budworm dynamics are governed by the differential equation (after a bunch of scaling)

$$\frac{dv}{dt} = \sigma v(1-v) - \frac{v^2}{\kappa^2 + v^2} = f(v). \tag{30}$$

We would like to understand for what parameter values this has bistable behavior. To find double zeros of f, we calculate the resultant of the numerators of f and f', which is

$$R = k^2\sigma(-4k^5\sigma^6 - 8k^4\sigma^6 - 12k^4\sigma^5 - 4k^3\sigma^6 + 20k^3\sigma^5 - 12k^3\sigma^4 + k^2\sigma^4 - 4k^2\sigma^3), \tag{31}$$

where $k = \kappa^2$. The zero level surface of R is shown plotted in Fig. 1(Right).

To determine cusp point we first solve for σ as a function of v

$$\sigma = \frac{v^2}{(v(1-v))(\kappa^2 + v^2)} = g(v), \tag{32}$$

and look for double roots $g'(v) = 0$. The necessary resultant is

$$R_\sigma = -32k^3(27k-1)(k+1)^2 \tag{33}$$

so that the cusp occurs for $\kappa = \frac{1}{\sqrt{27}}$. Similarly, solving for k,

$$\kappa^2 = \frac{v^2}{\sigma v(1-v)} - v^2 = h(v), \tag{34}$$

and we look for double roots of $h'(v) = 0$. The corresponding resultant is

$$R_k = -8\sigma^3(8\sigma - 27), \tag{35}$$

from which we learn that the cusp is at $\sigma = \frac{27}{8}$.

2.2 Quorum Sensing

The equations are (after a bunch of rescaling)

$$\frac{du}{dt} = s_0 + \gamma\frac{u^2}{1+u^2} + v - u, \tag{36}$$

$$\frac{dv}{dt} = -v + \frac{\rho}{1-\rho}(u-v). \tag{37}$$

It is not hard to determine that the steady states solutions are given by

$$\rho = -\frac{\gamma u^2 + s_0 u^2 - u^3 + s_0 - u}{(u^2+1)u} \equiv f(u). \tag{38}$$

To find regions of different behavior we look for double zeros of $f'(u) = 0$ and double zeros of $f(u) = 0$. We find that double zeros of $f'(u)$ occur when

$$\gamma = 8s_0, \tag{39}$$

and double zeros of f occur when

$$4\gamma^3 s_0 + 12\gamma^2 s_0^2 + 12\gamma s_0^3 + 4s_0^4\gamma^2 - 20\gamma s_0 + 8s_0^2 + 4 = 0, \tag{40}$$

Furthermore, the cusp intersect the straight line at $s_0 = \frac{1}{\sqrt{27}}$.

Remark: This analysis can be applied to *any* rational system of equations to (first) reduce the system of polynomials to a single polynomial of one variable and then (second) understanding the bifurcation structure of the solutions as a function of parameters, using resultant analysis.

2.3 Oscillations; Hopf Bifurcations

Resultants can also be used to find Hopf bifurcation points. The idea is as follows. For a system of equations,

$$\frac{dX}{dt} = F(X, p), \tag{41}$$

steady states are found as solutions of a polynomial equations $F(X, p) = 0$. Hopf bifurcation points are those for which

$$det(\frac{\partial F}{\partial X} - i\omega I) = 0 \tag{42}$$

This corresponds to two more equations (take real and imaginery parts) which must be simultaneously zero. The resultant of these two is a single polynomial in X and p. Now eliminate X using resultants to find a polynomial in p, which gives the locations of Hopf points.

2.4 Brusselator equations

Look for Hopf bifurcations for the Brusselator equations

$$u' = a - (b + 1)u + vu^2, \qquad v' = bu - vu^2. \tag{43}$$

Hopf bifurcation resultant analysis gives a Hopf bifurcation when $R = 0$, where

$$R = a(-a^2 + b - 1). \tag{44}$$

2.5 Lorenz Equations

The Lorenz equations are

$$\frac{dx}{dt} = \sigma(y - x), \tag{45}$$

$$\frac{dy}{dt} = x(\rho - z) - y, \tag{46}$$

$$\frac{dz}{dt} = xy - \beta z. \tag{47}$$

It is an easy matter to show that steady solutions are $x = 0$, and $x^2 = \rho\beta$. Further, there is a Hopf bifurcation at the nontrivial steady state provided

$$\beta\rho + \beta\sigma - \rho\sigma + \sigma^2 + \rho + 3\sigma = 0. \tag{48}$$

(determined using the resultant).

2.6 Bazykin's Equations

$$\begin{aligned} x' &= x - \frac{xy}{1+ax} - bx^2, \\ y' &= -gy + \frac{xy}{1+ax} - dy^2, \end{aligned}$$

We can find the cusp bifurcation curve to be

$$R_{cusp} = a(ag + bg - 1) + 8b. \tag{49}$$

Clearly, this makes sense only if $g(a + b) < 1$.

Similarly, the resultant analysis yields that Hopf bifurcations occur for $R = 0$ where

$$\begin{aligned} R \equiv\ & a^3dg^3 + a^2bdg^3 + a^3d^2g + 3a^2bd^2g + 4a^2bdg^2 + 3ab^2d^2g + 4ab^2dg^2 + b^3d^2g - a^3d^2 \\ & -a^3dg - 3a^2bd^2 - 7a^2bdg - 4a^2bg^2 - 2a^2d^2g - 2a^2dg^2 - 3ab^2d^2 - 7ab^2dg - 4ab^2g^2 \\ & -4abd^2g - b^3d^2 - b^3dg - 2b^2d^2g + a^2d^2 + 2abd^2 - 2abdg + ad^2g + b^2d^2 + 6b^2dg \\ & +bd^2g + 4abd + 4abg + ad^2 + adg - 4b^2d - 4b^2g + bd^2 - bdg - 4bd - d^2 \end{aligned}$$

While this formula is complicated, it not difficult to make contour plots of this function

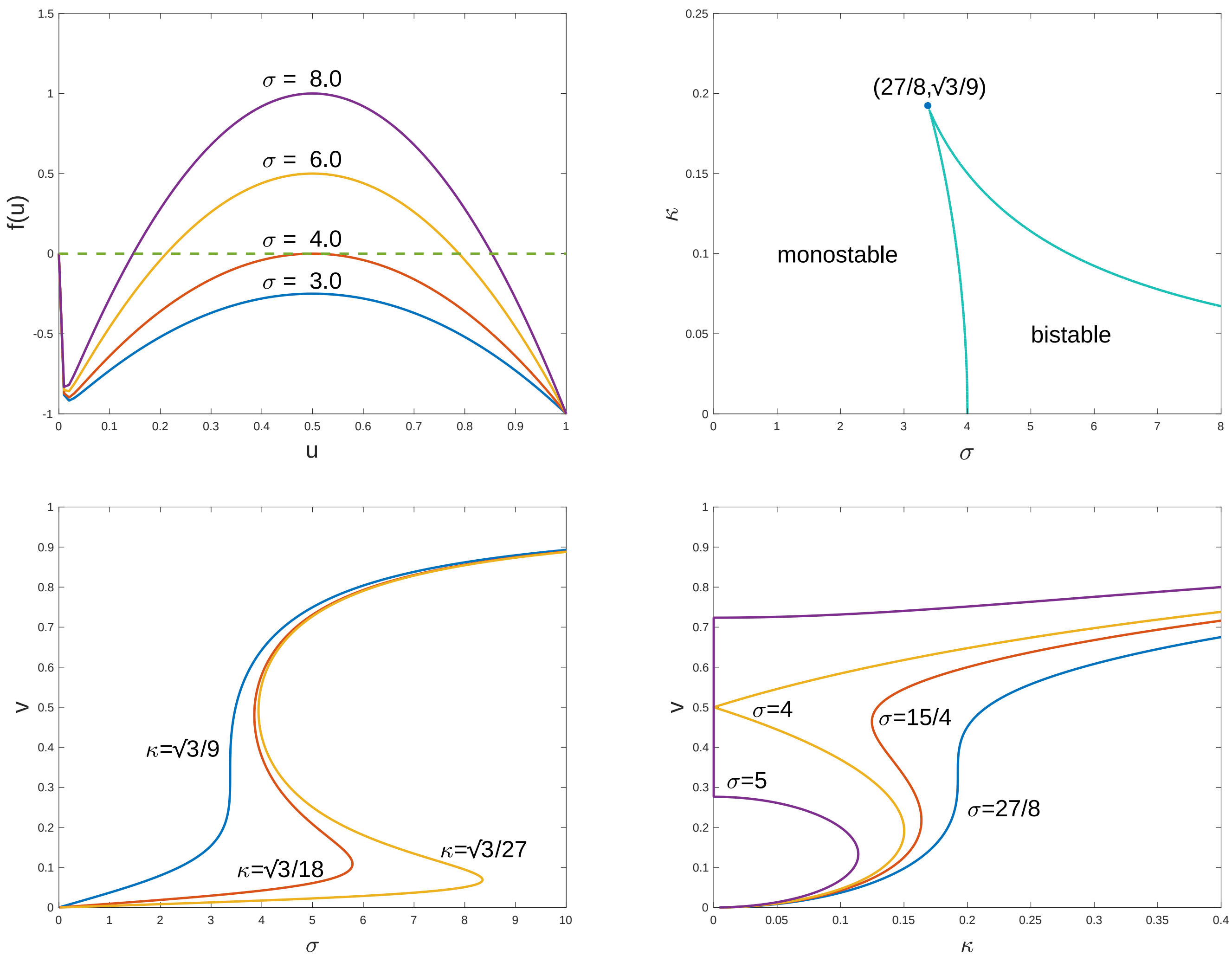

Figure 1: Top Left: Plot of $f(v)$ for $\kappa = 0.0031$ and for selected values of σ for Eqn. (30). Top Right: plot of the zero level set, $R(\kappa, \sigma) = 0$. Bottom Left: Plot of the steady solution v as a function of the parameter σ for several different values of κ, and Bottom Right: Plot of the steady solution v as a function of κ for several values of σ.

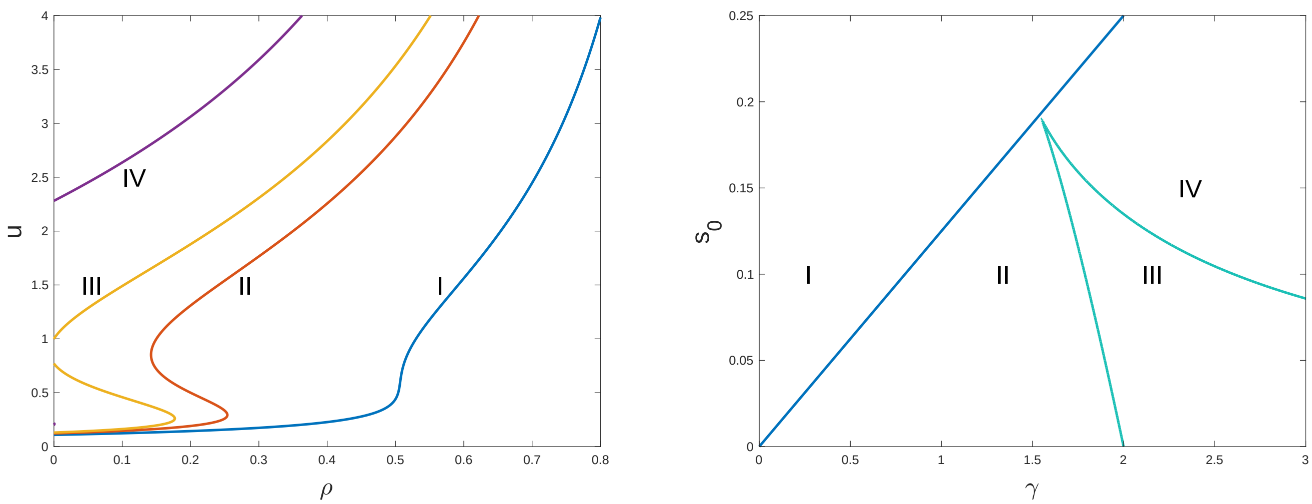

Figure 2: Left: Plot of the steady solution v as a function of the parameter ρ for several different values of γ with $s_0 = 0.1$ for Eqns. (36)-(37). Right: plot of the zero level set, $R(\gamma, s_0) = 0$.