Normal Form for the Fold Bifurcation

The purpose of these notes is to give a simple proof (simpler than Kuznetsov) that the normal form for the fold bifurcation is

$$\frac{dx}{dt} = a \pm x^2 \tag{1}$$

Theorem: If the function $f(y, a)$ is smooth and

1. $f(0, 0) = 0$
2. $f_y(0, 0) = 0$
3. $f_{yy}(0, 0) \neq 0$, $f_a(0, 0) \neq 0$,

then the flow $\frac{dy}{dt} = f(y, a)$ is topologically equivalent to the flow (**??**) in some sufficiently small neighborhood of the origin $y = 0, a = 0$.

The proof is in three steps. (Actually only two are necessary, but it is a bit easier to break it into three.) We only consider the case $f_{yy}(0, 0) < 0$, but the opposite case is equally easy.

Step 1: Suppose $f(\pm\sqrt{a}, a) = 0$ for all a, and $f_{yy}(0, 0) < 0$. Then the function

$$\mu(y, a) = \frac{a - y^2}{f(y, a)} \tag{2}$$

is a bounded positive function. It follows that the flow $\frac{dy}{dt} = f(y, a)$ is topologically equivalent to the flow $\frac{dx}{dt} = a - x^2$ since

$$a - x^2 = \mu(x, a) f(x, a) \tag{3}$$

Step 2: For $f(y, a)$ there is a function $y = y(a)$ so that $f_y(y(a), a) = 0$. This follows immediately from the Implicit Function Theorem.

Let $z = y - y(a)$. Then

$$\frac{dz}{dt} = g(z, a) = g_0(a) + g_2(a)z^2 + g_3(a)z^3 + \cdots. \tag{4}$$

Introduce a change of parameter $b = g_0(a)$ so that

$$\frac{dz}{dt} = G(z, b) = b + G_2(b)z^2 + G_3(b)z^3 + \cdots. \tag{5}$$

Step 3: If $G_2(b) < 0$, there is a linear change of variables $y = \alpha(b) + \beta(b)x$ with $\alpha = O(b), \beta = O(1)$, so that for $F(x, b) = G(\alpha(b) + \beta(b)x, b)$

$$F(\pm\sqrt{b}, b) = (G(\alpha(b) \pm \beta(b)\sqrt{b}), b) = 0 \tag{6}$$

for all sufficiently small b.

The proof follows easily from the Implicit Function Theorem. In fact, the transformation is (use Maple; see the Maple code at the bottom of these notes)

$$y = \frac{G_3}{2G_2^2} b + \frac{x}{\sqrt{-G_2}} \tag{7}$$

to leading order in b.

This same technique can be used to prove the following:

Theorem: Suppose $f(y, a) = ay + y^2 g(y, a)$ and $g(0, 0) \neq 0$. Then the normal form for the flow

$$\frac{dy}{dt} = f(y, a) \tag{8}$$

is

$$\frac{dx}{dt} = ax \pm x^2. \tag{9}$$

Theorem: Suppose $f(y, a) = ay + y^3 g(y, a)$ and $g(0, 0) \neq 0$. Then the normal form for the flow

$$\frac{dy}{dt} = f(y, a) \tag{10}$$

is

$$\frac{dx}{dt} = ax \pm x^3. \tag{11}$$

Here is the Maple code to do Step 3 (with a slightly different notation):

```
# Suppose f(y,a)  = a - f2^2(a) *y^2 + y^2*f3(y,a);  The following maple
# code determines the transformation y = c(a) + d(a) x so that
#g(x,a) = f(c(a) + d(a) x,a) has roots at x = +/- sqrt(a)

f:=b^2 -f2^2*y^2 + f3*y^3 + f4*y^4 + f5*y^5;
#here the use of b^2 rather than a and f2^2 rather than f2  to
#ensure that these quantities are positive.

#specify n
n:= 2;

#The unknown change of coordinate is:
y:= sum('c||k*b^(2*k) + d||k*b^(2*k)*x','k'=0..n);
c0:=0;

eq1:=collect(subs(x=b,f),b);
eq2:=collect(subs(x=-b,f),b);

eqa:=collect(eq1+eq2,b);
eqb:=collect(eq1-eq2,b);

coeff(eqa,b,2);
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d0:=1/f2;

c1:=solve(coeff(eqb,b,3),c1);

d1:=solve(coeff(eqa,b,4),d1);

c2:=solve(coeff(eqb,b,5),c2);

d2:=solve(coeff(eqa,b,6),d2);
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