Math 6740
Homework Exercises

1 Homotopy, Continuation and Branching Problems; Due Feb. 12, 2019

1. Use resultant analysis and XPPAUT to determine the root structure (i.e., bifurcation diagram) for the polynomial $f(x) = x^5 - x^3 + ax + b$. How many cusps are there and what are their locations in a, b parameter space?

2. Spruce budworm dynamics are governed by the differential equation (after a bunch of scaling)
$$\frac{dv}{dt} = \sigma v(1-v) - \frac{v^2}{\kappa^2 + v^2} = f(v).$$
Find the bifurcation diagram for this differential equation using XPPAUT, and compare it to an analytical expression for the location in parameter space of the double roots of $f(v)$.

3. Find solutions of the system of equations
$$\frac{dx}{dt} = \frac{\mu}{1+y^4} - x, \tag{1}$$
$$\frac{dy}{dt} = \frac{\mu}{1+x^4} - y, \tag{2}$$
$$\tag{3}$$
using XPPAUT. Find the value of μ (analytically) at which there is a pitchfork bifurcation from the symmetric solution $x = y = x_{eq}(\mu)$.

4. Find solutions of the system of equations
$$\frac{dx}{dt} = \frac{\mu}{1+y^4} - x, \tag{4}$$
$$\frac{dy}{dt} = \frac{\mu}{1+z^4} - y, \tag{5}$$
$$\frac{dz}{dt} = \frac{\mu}{1+x^4} - z, \tag{6}$$
using XPPAUT. Find the value of μ (analytically) at which there is a Hopf bifurcation for the symmetric solution $x = y = z = x_{eq}(\mu)$.

5. (a) Find the bifurcation diagram for Bazykin's equations

$$\begin{aligned} x' &= x - \frac{xy}{1+ax} - bx^2, \\ y' &= -gy + \frac{xy}{1+ax} - dy^2, \end{aligned}$$

as a function of the parameter d, $0 < d < 0.6$ and with $g = 1.0$, $b = 0.01$, $a = 0.3$. Draw phase portraits for each qualitatively different parameter range.

(b) Find analytical and numerical curves for the locus of limit points and the locus of Hopf points in the $d - g$ plane, keeping b and a fixed, and show that these (analytical and numerical) agree.

6. Find the behavior of solutions of the Hammerstein integral equation

$$u(x) - \lambda \int_0^1 k(x,y) f(u(y),y)dy = 0,$$

where $k(x,y) = \alpha(x)\alpha(y)$, and $f(u,y) = \frac{1}{2}(1+u^2)$. Assume that $\alpha(x) > 0$.

2 Topological Equivalence and Normal Forms; Due March 19, 2014

1. Determine which of the following linear systems has a structurally stable equilibrium at the origin, and sketch its phase portrait. Include in the sketch of the phase portrait, the stable, unstable or center manifolds (i.e., the tangent spaces associated with eigenvalues with negative, positive or zero, respectively, real parts).

   (a)

$$\begin{aligned}\dot{x} &= x - 2y,\\ \dot{y} &= -2x + 4y,\end{aligned}$$

   (b)

$$\begin{aligned}\dot{x} &= 2x + y,\\ \dot{y} &= -x,\end{aligned}$$

   (c)

$$\begin{aligned}\dot{x} &= x + 2y,\\ \dot{y} &= -x - y.\end{aligned}$$

2. Write the system of equations

$$\begin{aligned}\frac{dx}{dt} &= \lambda x - y\\ \frac{dy}{dt} &= \lambda y + x - x^2 y,\end{aligned}$$

   as a single equation in the complex variable $z = x + iy$, and compute the normal form for this equation. Describe the bifurcations of this system as a function of the parameter λ. Verify your results using XPP.

3. Find the direction of Hopf bifurcations in the parameter b for the Brussleator equarions

$$u' = a - (b+1)u + vu^2, \qquad v' = bu - vu^2.$$

4. Find the direction of bifurcation for the differential equation

$$u'' - \lambda u' + u^2 u' + u - \frac{1}{3}u'^3 = 0.$$

5. Use phase plane arguments to determine the global Hopf bifurcation curve for the equation

$$x'' + ax' + x - x^3 = 0.$$

3 Due April 9, 2019

1. Find (numerically) the rotation number as a function of the parameter μ, $0 < \mu < 1$ for the *sine circle map*

$$x_{n+1} = \mathrm{mod}(\mu + x_n + a\sin(2\pi x_n), 1),$$

for $a = 0.155$. Find a relationship between μ and a for which the rotation number is 0, for $0 < 2\pi a < 1$.

2. Consider the discrete-time system

$$\begin{aligned} x_{k+1} &= \alpha x_k(1 - x_k) - x_k y_k \\ y_{k+1} &= \frac{1}{\beta} x_k y_k. \end{aligned}$$

Show that a nontrivial fixed point undergoes a Neimark-Sacker bifurcation on a curve in the (α, β) plane, and determine the direction of the closed invariant-curve bifurcation (a numerical simulation for this suffices).

3. Find the center manifold and determine the asymptotic stability of the origin for the system of equations

$$\frac{dx}{dt} = y, \tag{7}$$

$$\frac{dy}{dt} = -x - xv, \tag{8}$$

$$\frac{dv}{dt} = -v + \alpha x^2, \tag{9}$$

as a function of the parameter α.

4 Codimension Two Bifurcations; Due May 3, 2011

1. Show that there is a Bautin codimension 2 bifurcation at $\lambda = \alpha = 0$ for the differential equation
$$\frac{d^2u}{dt^2} - \lambda\frac{du}{dt} + u + u^2\frac{du}{dt} + (\alpha - \frac{1}{3})(\frac{du}{dt})^3 - \frac{u^3}{2} = 0.$$
Determine the (approximate) locus of saddle-node-periodic solutions (SNP's) in the $\lambda - \alpha$ plane for this equation.

2. Show that the system
$$\frac{dx}{dt} = x - xy + 1 \tag{10}$$
$$\frac{dy}{dt} = ay + bx^2 \tag{11}$$
has a Takens-Bogdonov codimension 2 bifurcation point. Sketch the phase portraits of the system in the vicinity of this point.