1 Homotopy, Continuation and Nonlinear Eigenvalue Problems; Due Feb. 6, 2014

1. Use homotopy and XPPAUT to find all the real roots of the polynomial $f(x) = x^5 - 3x^4 - 20x^3 + 20x^2 - 20x - 20$. Plot the locus of solutions as a function of your homotopy parameter.

2. (a) Use XPPAUT to find nontrivial solutions of $u'' + \lambda ue^u = 0$ subject to boundary conditions $u(0) = u(1) = 0$. Find the largest value of $\lambda$ for which positive solutions exist.
   (b) Find analytical expressions for the first two (nontrivial) solution branches of the boundary value problem $u'' + \lambda ue^u = 0$ subject to boundary conditions $u(0) = u(1) = 0$. Verify that the bifurcation diagram found by XPPAUT is correct.

3. Compute the one-parameter bifurcation diagram for the Brusselator equations
   $$u' = a - (b + 1)u + vu^2, \quad v' = bu - vu^2$$
   with $a = 1$, $b = 1.5$, $u = 1$, $v = 1.5$, treating $b$ as the bifurcation parameter $0 \leq b \leq 4$. Find the phase portrait for $b = 1$ and $b = 3$. What are the differences between solutions for these parameter values and how are these differences evident from the bifurcation diagram?

4. Find the bifurcation diagram for Bazykin’s equations
   $$x' = x - \frac{xy}{1+ax} - bx^2, \quad y' = -gy + \frac{xy}{1+ax} - dy^2,$$
   as a function of the parameter $d$, $0 < d < 0.6$ and with $g = 1.0$, $b = 0.01$, $a = 0.3$. Draw phase portraits for each qualitatively different parameter range.

5. Find the bifurcation diagram for the Lorenz equations
   $$x' = -y^2 - z^2 - ax + aF, \quad y' = xy - bxz - y + G, \quad z' = bxy + xz - z,$$
as a function of the parameter $F$, for $a = 0.25$, $b = 4$, $G = 0.5$. Find the locus of limit points and locus of Hopf points in the $F - G$ parameter plane.

2  Topological Equivalence; Due March 18, 2014

1. Determine which of the following linear systems has a structurally stable equilibrium at the origin, and sketch its phase portrait. Include in the sketch of the phase portrait, the stable, unstable or center manifolds (i.e., the tangent spaces associated with eigenvalues with negative, positive or zero, respectively, real parts).

(a)
\[
\begin{align*}
\dot{x} &= x - 2y, \\
\dot{y} &= -2x + 4y,
\end{align*}
\]

(b)
\[
\begin{align*}
\dot{x} &= 2x + y, \\
\dot{y} &= -x,
\end{align*}
\]

(c)
\[
\begin{align*}
\dot{x} &= x + 2y, \\
\dot{y} &= -x - y.
\end{align*}
\]

3  Normal Forms; Due April 8, 2014

1. Write the system
\[
\begin{align*}
\frac{dx}{dt} &= \lambda x - y - xy + 2y^2, \\
\frac{dy}{dt} &= \lambda y + x - x^2 y,
\end{align*}
\]
as a single equation in the complex variable $z = x + iy$, and compute the normal form for this equation. Describe the bifurcations of this system as a function of the parameter $\lambda$. Verify your results using XPP.

2. Show that the following systems have Hopf bifurcations and determine the direction of bifurcation analytically and verify this using XPP.
(a) 
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \alpha y + x^2 + xy + y^2.
\]

(b) 
\[
\frac{dx}{dt} = \alpha(1 - xy^2 + A(y - 1)), \quad \frac{dy}{dt} = xy^2 - y.
\]

3. Consider the discrete-time system

\[
x_{k+1} = \alpha x_k(1 - x_k) - x_ky_k \\
y_{k+1} = \frac{1}{\beta} x_ky_k.
\]

Show that a nontrivial fixed point undergoes a Neimark-Sacker bifurcation on a curve in the \((\alpha, \beta)\) plane, and determine the direction of the closed invariant-curve bifurcation (a numerical simulation for this suffices).