

Math 6740  
Homework Exercises

## 1 Homotopy, Continuation and Nonlinear Eigenvalue Problems; Due Feb. 22, 2011

1. Use homotopy and XPPAUT to find all the real roots of the polynomial  $f(x) = x^5 - 3x^4 - 20x^3 + 20x^2 - 20x - 20$ . Plot the locus of solutions as a function of your homotopy parameter.
2. (a) Use XPPAUT to find the solutions of  $u'' + \lambda ue^u = 0$  subject to boundary conditions  $u(0) = u(1) = 0$ . Find the largest value of  $\lambda$  for which solutions exist.  
(b) Find analytical expressions for the first two solution branches of the boundary value problem

$$u'' + \lambda ue^u = 0$$

subject to boundary conditions  $u(0) = u(1) = 0$ . Verify that the bifurcation diagram found by XPPAUT is correct.

3. Compute the one-parameter bifurcation diagram for the Brusselator equations

$$u' = a - (b + 1)u + vu^2, \quad v' = bu - vu^2$$

with  $a = 1$ ,  $b = 1.5$ ,  $u = 1$ ,  $v = 1.5$ , treating  $b$  as the bifurcation parameter  $0 \leq b \leq 4$ .

4. Discuss the behavior of solutions of the Hammerstein integral equation

$$u(x) - \lambda \int_0^1 k(x, y)f(u(y), y)dy = 0$$

where  $k(x, y) = \alpha(x)\alpha(y)$ , and  $f(u, y) = \frac{1}{2}(1 + u^2)$ .

5. Further reading: read the paper by I. Stakgold, Branching of solutions of nonlinear equations, SIAM Rev, 13, 289-332 (1971).

## 2 Normal Forms; Due March 29, 2011

1. Write the system

$$\frac{dx}{dt} = \lambda x - y - xy + 2y^2 \tag{1}$$

$$\frac{dy}{dt} = \lambda y + x - x^2y, \tag{2}$$

as a single equation in the complex variable  $z = x + iy$ , and compute the normal form for this equation. Describe the bifurcations of this system as a function of the parameter  $\lambda$ .

2. Show that the following systems have Hopf bifurcations in the parameter  $\alpha$  and determine the direction of bifurcation.

(a)

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \alpha y + x^2 + xy + y^2,$$

(b)

$$\frac{dx}{dt} = \alpha(1 - xy^2 + A(y - 1)), \quad \frac{dy}{dt} = xy^2 - y.$$

3. Consider the discrete-time system

$$x_{k+1} = \alpha x_k(1 - x_k) - x_k y_k \tag{3}$$

$$y_{k+1} = \frac{1}{\beta} x_k y_k \tag{4}$$

Show that a nontrivial fixed point undergoes a Neimark-Sacker bifurcation on a curve in the  $(\alpha, \beta)$  plane, and compute the direction of the closed invariant-curve bifurcation.

4. Find the center manifold equations for the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x - xv, \quad \frac{dv}{dt} = -v + \alpha x^2$$

Determine the stability of the origin.

### 3 Codimension One Bifurcations; Due May 3, 2011

1. Determine the nature of the homoclinic (or heteroclinic) bifurcations in the systems

(a)

$$\frac{dx}{dt} = y + \alpha x, \quad \frac{dy}{dt} = -x^3 + x + xy,$$

(b)

$$\frac{dx}{dt} = \alpha + 2xy, \quad \frac{dy}{dt} = 1 + x^2 - y^2$$

2. Consider the predator-prey system

$$\frac{dx}{dt} = x - \frac{xy}{1 + \alpha x} - \epsilon x^2 \quad (5)$$

$$\frac{dy}{dt} = -\gamma y + \frac{xy}{1 + \alpha x} - \delta y^2. \quad (6)$$

- (a) Find the curves of Hopf and fold bifurcation points in the  $(\alpha, \delta)$ -plane ( $\epsilon = 0$ ).  
Show that they intersect at a Bogdanov-Takens point.
- (b) Find the  $(\alpha, \delta)$  coordinates of the cusp bifurcation points.  
Hint: the cusp points are triple roots of a polynomial.
- (c) What other bifurcation curves do you expect? Why?