1 Facilitated Diffusion - Corner Layer Analysis

The equations are

$$\epsilon_1 \sigma_{xx} = \sigma(1-u) - u = -\epsilon_2 u_{xx} \tag{1}$$

with boundary conditions

$$\sigma(0) = \sigma_0, \qquad \sigma(1) = \sigma_1, \qquad u'(0) = u'(1) = 0. \tag{2}$$

There is a conservation law

$$\epsilon_1 \sigma_{xx} + \epsilon_2 u_{xx} = 0, \tag{3}$$

so that

$$\epsilon_1 \sigma_x + \epsilon_2 u_x = \epsilon_1 J, \tag{4}$$

and

$$\epsilon_1 \sigma + \epsilon_2 u = \epsilon_1 J x + \epsilon_1 \sigma_0 + \epsilon_2 u(0). \tag{5}$$

Thus we solve for σ and substitute into (1) to find

$$\epsilon_2 u_{xx} = u - (Jx + \sigma_0 + \gamma u(0) - \gamma u)(1-u), \tag{6}$$

where $\gamma = \frac{\epsilon_2}{\epsilon_1}$.

The outer solution is straightforward. We set $\epsilon_1 = \epsilon_2 = 0$ in (1) and observe that

$$\sigma(1-u) - u = 0, \tag{7}$$

so that

$$\sigma = \frac{u}{1-u}, \tag{8}$$

or

$$u = \frac{\sigma}{1+\sigma}, \tag{9}$$

from which we determine

$$u(0) = \frac{\sigma_0}{1+\sigma_0}, \qquad u(1) = \frac{\sigma_1}{1+\sigma_1} \tag{10}$$

so that

$$\epsilon_1 J = \epsilon_1 \sigma_1 + \epsilon_2 u(1) - \epsilon_1 \sigma_0 - \epsilon_2 u(0) = \epsilon_1(\sigma_1 - \sigma_0) + \epsilon_2\left(\frac{\sigma_1}{1+\sigma_1} - \frac{\sigma_0}{1+\sigma_0}\right). \tag{11}$$

We can also determine the outer solution as a solution of the quadratic equation

$$u - (Jx + \sigma_0 + \gamma u(0) - \gamma u)(1-u) = 0. \tag{12}$$

It follows that the derivative of u satisfies

$$(1 + \gamma(1-u) + (Jx + \sigma_0 + \gamma u(0) - \gamma u))u'(x) - J(1-u) = 0, \tag{13}$$

so that at $x = 0$

$$u'(0) = \frac{J(1-u(0))}{(1+\gamma(1-u(0)) + \sigma_0)} = \frac{1}{\alpha^2} J(1-u(0)), \tag{14}$$

where $\alpha^2 = (1 + \sigma_0 + \gamma(1 - U_0))$, which is not zero.

We need a corner layer expansion in order to satisfy the boundary conditions for u. For the corner layer near $x = 0$, we set $y = \frac{x}{\epsilon_2^{1/2}}$ to find

$$u_{yy} = u - (\epsilon_2^{1/2} Jy + \sigma_0 + \gamma u(0) - \gamma u)(1 - u). \tag{15}$$

We expand $u = U(y)$ as

$$U(y) = U_0(y) + \epsilon_2^{1/2} U_1(y) + \cdots, \tag{16}$$

and the resulting hierarchy of equations is

$$U_0'' = U_0 - (\sigma_0 + \gamma U_0(0) - \gamma U_0)(1 - U_0), \tag{17}$$

$$U_1'' = U_1 - (Jy + \gamma U_1(0) - \gamma U_1)(1 - U_0) + (\sigma_0 + \gamma U_0(0) - \gamma U_0)U_1. \tag{18}$$

The first of these we solve with $U_0'(0) = 0$, and the only solution is the constant solution $U_0 = U(0) = u_0$. The second order equation can be written as

$$U_1'' - \alpha^2 U_1 = -(Jy + \gamma U_1(0))(1 - U_0). \tag{19}$$

The solution is

$$U_1(y) = A \exp(-\alpha y) + \frac{1}{\alpha^2}(Jy + \gamma U_1(0))(1 - U_0). \tag{20}$$

For consistency,

$$A = U_1(0) - \frac{1}{\alpha^2}\gamma U_1(0)(1 - U_0), \tag{21}$$

and $U'(0) = 0$ implies that

$$U_1'(0) = -\alpha A + \frac{1}{\alpha^2} J(1 - U_0) = 0 \tag{22}$$

so that

$$A = \frac{1}{\alpha^3} J(1 - U_0), \qquad U_1(0) = \alpha \frac{1 - U_0}{(\alpha^2 - \gamma(1 - U_0)}. \tag{23}$$

Now let's see if these match. (They had better since there are no free parameters to adjust.) We introduce the intermediate variable $z = \frac{x}{\epsilon_2^{1/4}}$, so that $x = \epsilon_2^{1/4} z$ and $y = \frac{z}{\epsilon_2^{1/4}}$. We must examine

$$\begin{aligned}
\lim_{\epsilon_2 \to 0} \frac{u_0(x) - U(y)}{\epsilon_2^{1/4}} &= \lim_{\epsilon_2 \to 0} \frac{u_0(\epsilon_2^{1/4} z) - U(\frac{z}{\epsilon_2^{1/4}})}{\epsilon_2^{1/4}} \\
&= \lim_{\epsilon_2 \to 0} \frac{u_0'(0)(\epsilon_2^{1/4} z) - \epsilon_2^{1/2} A \exp(-\alpha \frac{z}{\epsilon_2^{1/4}}) - \epsilon_2^{1/2} \frac{1}{\alpha^2}(J \frac{z}{\epsilon_2^{1/4}} + \gamma U_1(0))(1 - U_0)}{\epsilon_2^{1/4}} \\
&= \lim_{\epsilon_2 \to 0} \frac{u_0'(0)(\epsilon_2^{1/4} z) - \frac{1}{\alpha^2} z \epsilon_2^{1/4} J(1 - u_0)}{\epsilon_2^{1/4}} \\
&= z(u_0'(0) - \frac{1}{\alpha^2} J(1 - u_0)) = 0,
\end{aligned}$$

as hoped. The end result of this is that the composite solution is given by

$$u(x) = u_0(x) + \epsilon_2^{1/2} A \exp(\frac{x}{\epsilon}), \tag{24}$$

which takes care of the corner layer at $x = 0$. A similar analysis implies a similar corner layer at $x = 1$, yielding a solution of the form

$$u(x) = u_0(x) + \epsilon_2^{1/2} \frac{u_0'(0)}{\alpha} \exp(-\alpha \frac{x}{\epsilon_2^{1/2}}) + \epsilon_2^{1/2} \frac{u_0'(1)}{\beta} \exp(\beta \frac{1-x}{\epsilon_2^{1/2}}) + \cdots, \tag{25}$$