1 Facilitated Diffusion - Corner Layer Analysis

The equations are

\[ \epsilon_1 \sigma_{xx} = \sigma(1 - u) - u = -\epsilon_2 u_{xx} \tag{1} \]

with boundary conditions

\[ \sigma(0) = \sigma_0, \quad \sigma(1) = \sigma_1, \quad u'(0) = u'(1) = 0. \tag{2} \]

There is a conservation law

\[ \epsilon_1 \sigma_{xx} + \epsilon_2 u_{xx} = 0, \tag{3} \]

so that

\[ \epsilon_1 \sigma_x + \epsilon_2 u_x = \epsilon_1 J, \tag{4} \]

and

\[ \epsilon_1 \sigma + \epsilon_2 u = \epsilon_1 J x + \epsilon_1 \sigma_0 + \epsilon_2 u(0). \tag{5} \]

Thus we solve for \( \sigma \) and substitute into (1) to find

\[ \epsilon_2 u_{xx} = u - (J x + \sigma_0 + \gamma u(0) - \gamma u)(1 - u), \tag{6} \]

where \( \gamma = \frac{\epsilon_2}{\epsilon_1} \).

The outer solution is straightforward. We set \( \epsilon_1 = \epsilon_2 = 0 \) in (1) and observe that

\[ \sigma(1 - u) - u = 0, \tag{7} \]

so that

\[ \sigma = \frac{u}{1 - u}, \tag{8} \]

or

\[ u = \frac{\sigma}{1 + \sigma}, \tag{9} \]

from which we determine

\[ u(0) = \frac{\sigma_0}{1 + \sigma_0}, \quad u(1) = \frac{\sigma_1}{1 + \sigma_1} \tag{10} \]

so that

\[ \epsilon_1 J = \epsilon_1 \sigma_1 + \epsilon_2 u(1) - \epsilon_1 \sigma_0 - \epsilon_2 u(0) = \epsilon_1 (\sigma_1 - \sigma_0) + \epsilon_2 \left( \frac{\sigma_1}{1 + \sigma_1} - \frac{\sigma_0}{1 + \sigma_0} \right). \tag{11} \]

We can also determine the outer solution as a solution of the quadratic equation

\[ u - (J x + \sigma_0 + \gamma u(0) - \gamma u)(1 - u) = 0. \tag{12} \]

It follows that the derivative of \( u \) satisfies

\[ (1 + \gamma(1 - u) + (J x + \sigma_0 + \gamma u(0) - \gamma u))u'(x) - J(1 - u) = 0, \tag{13} \]

so that at \( x = 0 \)

\[ u'(0) = \frac{J(1 - u(0))}{(1 + \gamma(1 - u(0)) + \sigma_0)} = \frac{1}{\alpha^2} J(1 - u(0)), \tag{14} \]
where \( \alpha^2 = (1 + \sigma_0 + \gamma(1 - U_0)) \), which is not zero.

We need a corner layer expansion in order to satisfy the boundary conditions for \( u \). For the corner layer near \( x = 0 \), we set \( y = \frac{x}{\epsilon_2^2} \) to find

\[
  u_{yy} = u - (\epsilon_2^{1/2} J y + \sigma_0 + \gamma u(0) - \gamma u)(1 - u). \tag{15}
\]

We expand \( u = U(y) \) as

\[
  U(y) = U_0(y) + \epsilon_2^{1/2} U_1(y) + \cdots, \tag{16}
\]

and the resulting hierarchy of equations is

\[
  U_0'' = U_0 - (\sigma_0 + \gamma U_0(0) - \gamma U_0)(1 - U_0), \tag{17}
\]

\[
  U_1'' = U_1 - (J y + \gamma U_1(0) - \gamma U_1)(1 - U_0) + (\sigma_0 + \gamma U_0(0) - \gamma U_0) U_1. \tag{18}
\]

The first of these we solve with \( U_0'(0) = 0 \), and the only solution is the constant solution \( U_0 = U(0) = u_0 \). The second order equation can be written as

\[
  U_1'' - \alpha^2 U_1 = -(J y + \gamma U_1(0))(1 - U_0). \tag{19}
\]

The solution is

\[
  U_1(y) = A \exp(-\alpha y) + \frac{1}{\alpha^2} (J y + \gamma U_1(0))(1 - U_0). \tag{20}
\]

For consistency,

\[
  A = U_1(0) - \frac{1}{\alpha^2 \gamma U_1(0)(1 - U_0)}, \tag{21}
\]

and \( U'(0) = 0 \) implies that

\[
  U_1'(0) = -\alpha A + \frac{1}{\alpha^2} J (1 - U_0) = 0 \tag{22}
\]

so that

\[
  A = \frac{1}{\alpha^3} J (1 - U_0), \quad U_1(0) = \alpha \frac{1 - U_0}{(\alpha^2 - \gamma(1 - U_0))}. \tag{23}
\]

Now let’s see if these match. (They had better since there are no free parameters to adjust.) We introduce the intermediate variable \( z = \frac{x}{\epsilon_2^2} \), so that \( x = \epsilon_2^{1/4} z \) and \( y = \frac{z}{\epsilon_2^2} \). We must examine

\[
  \lim_{\epsilon_2 \to 0} \frac{u_0(x) - U(y)}{\epsilon_2^{1/4}} = \lim_{\epsilon_2 \to 0} \frac{u_0(\epsilon_2^{1/4} z) - U\left(\frac{z}{\epsilon_2^{2/4}}\right)}{\epsilon_2^{1/4}}
  = \lim_{\epsilon_2 \to 0} \frac{u_0'(0)(\epsilon_2^{1/4} z) - \epsilon_2^{1/2} A \exp(-\alpha \frac{z}{\epsilon_2^{1/4}}) - \epsilon_2^{1/2} \frac{1}{\alpha^2} (J \frac{z}{\epsilon_2^{1/4}} + \gamma U_1(0))(1 - U_0)}{\epsilon_2^{1/4}}
  = \lim_{\epsilon_2 \to 0} \frac{u_0'(0)(\epsilon_2^{1/4} z) - \frac{1}{\alpha^2} \gamma \epsilon_2^{1/4} J (1 - u_0)}{\epsilon_2^{1/4}}
  = \frac{z(u_0'(0) - \frac{1}{\alpha^2} J (1 - u_0))}{\epsilon_2^{1/4}} = 0,
\]

where \( \alpha^2 = (1 + \sigma_0 + \gamma(1 - U_0)) \), which is not zero.
as hoped. The end result of this is that the composite solution is given by

$$u(x) = u_0(x) + \epsilon^1/2 A \exp\left(\frac{x}{\epsilon}\right),$$  \hspace{1cm} (24)

which takes care of the corner layer at $x = 0$. A similar analysis implies a similar corner layer at $x = 1$, yielding a solution of the form

$$u(x) = u_0(x) + \epsilon^1/2 \frac{u_0'(0)}{\alpha} \exp\left(-\alpha \frac{x}{\epsilon^1/2}\right) + \epsilon^1/2 \frac{u_0'(1)}{\beta} \exp\left(\beta \frac{1-x}{\epsilon^1/2}\right) + \cdots,$$  \hspace{1cm} (25)