Math 6730
Homework Exercises - Solutions

1 Regular Perturbation Theory

(due Sept. 22, 2015)

1. Find a two-term asymptotic expansion, for small ϵ, for all solutions of the following equations:

   (a)
   $$\epsilon x^3 - 3x + 1 = 0 \tag{1}$$

   (b)
   $$x^{2+\epsilon} = \frac{1}{x + 2\epsilon} \tag{2}$$

   (c)
   $$xe^{-x} = \epsilon \tag{3}$$

   (d)
   $$x^3 - x^2 - \epsilon = 0 \tag{4}$$

2. Find a two-term asymptotic expansion, for small ϵ, of the solution of the following boundary value problems:

   (a)
   $$y'' + y + \epsilon y^3 = 0, \qquad y(0) = 0, \qquad y(\frac{\pi}{2}) = 1. \tag{5}$$

   (b)
   $$y'' - y = 0, \qquad y(0) = 0, \qquad y(1+\epsilon) = 1 \tag{6}$$

   Hint: Use a change of variables to transform this to a problem on a fixed domain (independent of ϵ).

3. Suppose that a one-dimensional cell that is 10 μm long and the voltage potential is held fixed at both ends (so that $\Phi = 0$ at the ends). Suppose that the cell contains approximately equal amounts (say 400mM) of sodium and chloride, but suppose there is a slight excess of sodium. How much excess sodium is there if the total difference between maximal and minimal potential in the cell is 0.1mV? Plot the (approximate) distribution of potential and ion concentrations for both species.

   Hint: Derive the Poisson-Boltzman equation as follows: According to the Poisson equation,
   $$\epsilon \nabla^2 \phi = -qN_a \sum_i z_i c_i \tag{7}$$

where ϵ is the dielectric constant of the medium, N_a is Avagadro's number, q is the unit electric charge, c_i and z_i are the concentration and unit charge for the i^{th} species. According to the Nernst-Planck equation,

$$0 = \nabla c_i + \frac{z_i F}{RT} c_i \nabla \phi. \tag{8}$$

Use this information to derive the Poisson-Boltzmann equation in dimensionless units

$$\nabla^2 \Phi = -\beta \sum_i z_i \alpha_i \exp(-z_i \Phi), \tag{9}$$

where

$$\beta = q \frac{L^2 F N_A}{\epsilon RT} C_0, \tag{10}$$

and C_0 is a characteristic concentration, L is the length of a one dimensional domain. What is α_i? For parameter values, use that $\epsilon = 6.938 \times 10^{-10}\mathrm{C}^2/\mathrm{Nm}^2$ at 25C, $q = 1.6 \times 10^{-19}$C(oulombs), $\frac{RT}{F} = 25.8$mV. How large is β for this cell?

Use that the net charge

$$\delta = \sum_i z_i \alpha_i. \tag{11}$$

is a small dimensionless parameter to find the asymptotic solution of the problem.

4. Approximate the time it takes for a projectile whose terminal velocity is V_T with height $y(t)$ above the earth, with $y(0) = 0$ and $y_t(0) = V_0$, to return to earth, assuming that $\frac{V_0^2}{gR}$ and $\frac{V_0}{V_T}$ are small. What is the effect of drag on the return time?
   Hint: Use the equation

$$y_{tt} = -\frac{g}{(1+\frac{y}{R})^2} - \frac{gR}{V_T(R+y)} y_t. \tag{12}$$

   to account for viscous atmospheric drag.

1.1 Solutions

1. (a) There are three solutions,

$$x = \frac{1}{3} + \frac{1}{81}\epsilon + \frac{1}{729}\epsilon^2 + O(\epsilon^3), \tag{13}$$

   and

$$x = \pm\sqrt{\frac{3}{\epsilon}} - \frac{1}{6} \mp \frac{\sqrt{3}}{72}\sqrt{\epsilon} + O(\epsilon). \tag{14}$$

   (b) Take the logarithm of both sides of the equation to rewrite the equation as

$$(3+\epsilon)\ln x + \ln(1 + \frac{2\epsilon}{x}) + 2k\pi i. \tag{15}$$

A power series solution in ϵ of the form $x = x_0+\epsilon x_1+\epsilon^2 x_2+\cdots$ yields $x_0 = \exp(\frac{\pm i\pi}{3})$ or $x = 1$ and $x_1 = -\frac{2}{3} - \frac{x_0}{3}\ln(x_0)$, and $x_2 = \frac{1}{6x_0}(3x_1^2 + 4x_1 + 4) - \frac{x_1}{3}$.
For $k = 0$, this gives

$$x = 1 - \frac{2}{3}\epsilon + \frac{2}{3}\epsilon^2 + O(\epsilon^3). \tag{16}$$

For $k = \pm 1$, the answer is tedious to write out.

(c) To find an asymptotic expansion for

$$x\exp(-x) = \epsilon, \tag{17}$$

make a few changes of variables. Let $\gamma = \frac{-1}{\ln\epsilon}$ and $y = \gamma x$ and set $\eta = \gamma\ln\gamma$ and the equation becomes

$$y = 1 - \eta + \gamma\ln(y). \tag{18}$$

Clearly y is nearly 1. A relatively easy way to proceed is iteratively, i.e., let $y_n = F(y_{n-1})$ where $F(y) = 1 - \eta + \gamma\ln(y)$, so that

$$y_0 = 1, \tag{19}$$

$$y_1 = 1 - \eta, \tag{20}$$

$$y_2 = 1 - \eta + \gamma\ln(1-\eta), \tag{21}$$

$$y_3 = 1 - \eta + \gamma\ln(1 - \eta + \gamma\ln(1-\eta)), \tag{22}$$

and so on. However, this is not an asymptotic series, so to make it into one we now expand y_n as a Taylor series in γ and find

$$y \approx -\eta+\gamma\ln(1-\eta)+\frac{\ln(1-\eta)}{(1-\eta)}\gamma^2+\left(\frac{\ln(1-\eta)}{(1-\eta)^2} - \frac{1}{2}\frac{(\ln(1-\eta))^2}{(1-\eta)^2}\right)\gamma^3+O\left(\left(\frac{\ln(1-\eta)}{1-\eta}\right)^3\gamma^4\right) \tag{23}$$

This is an asymptotic series, but a strange one.

(d) There are three solutions

$$x = 1 + \epsilon - 2\epsilon^2 + O(\epsilon^3), \tag{24}$$

and

$$x = \pm i\sqrt{\epsilon} - \frac{1}{2}\epsilon \mp \frac{5}{8}i\sqrt{\epsilon^3}. \tag{25}$$

2. (a)

$$y(x) = \sin(x) + \epsilon\Big(\frac{1}{4}\sin(x) - \frac{1}{8}\sin(x)\cos^2(x) + \frac{3}{8}x\cos(x) + \frac{2}{8}\sin(x)\Big). \tag{26}$$

(b)

$$y(x) = \frac{\sinh(t)}{\sinh(1)} + \epsilon\Big(\frac{t\cosh(t)}{\sinh(1)} - \frac{\cosh(1)\sinh(t)}{\sinh^2(1)}\Big) + O(\epsilon^2), \tag{27}$$

where $t = \frac{x}{1+\epsilon}$.

3. Take $z_1 = 1$ and $z_2 = -1$ for sodium and chloride, respectively, and then $\delta = \alpha_2 - \alpha_1$. It follows that

$$\nabla^2\Phi = -\beta(-\alpha_1 \exp(-\Phi) + (\delta + \alpha_1)\exp(\Phi)). \tag{28}$$

Set $\Phi = \delta\Phi_1 + O(\delta^2)$ and find that to leading order in δ,

$$\nabla^2\Phi_1 + 2\beta\alpha_1\Phi_1 = -\beta \tag{29}$$

The solution of this equation is easy, being

$$\Phi_1 = -\frac{1}{2\alpha_1}\left(1 - \frac{\cosh(\sqrt{2\beta\alpha_1}(x - \frac{1}{2}))}{\cosh(\frac{1}{2}\sqrt{2\beta\alpha_1})}\right). \tag{30}$$

Now suppose that the maximum value of Φ is Φ^*. This implies that

$$\frac{\delta}{2\alpha_1} = -\Phi^*. \tag{31}$$

Now by conservation, it must be that

$$1 = \int_0^1 \alpha_2 \exp(\Phi)dx \approx \int_0^1 \alpha_2(1 + \delta\Phi_1)dx, \tag{32}$$

and since

$$\int_0^1 \Phi_1 dx = -\frac{1}{2\alpha_1}\left(1 - 2\frac{\tanh(\frac{1}{2}\sqrt{2\beta\alpha_1})}{\sqrt{2\beta\alpha_1}}\right). \tag{33}$$

$$1 = \alpha_2\left(1 - \delta\frac{1}{2\alpha_1}\left(1 - 2\frac{\tanh(\frac{1}{2}\sqrt{2\beta\alpha_1})}{\sqrt{2\beta\alpha_1}}\right)\right) \tag{34}$$

Now we use that β is very large to get

$$1 = \alpha_2\left(1 - \delta\frac{1}{2\alpha_1}\right) = \alpha_2\left(1 + \Phi^*\right). \tag{35}$$

Similarly, total sodium (in dimensionless units) is

$$1 + \Delta_{Na} = \alpha_1\left(1 + \delta\frac{1}{2\alpha_1}\right) = \alpha_1\left(1 - \Phi^*\right) \tag{36}$$

We can now use the three equations (31,35,36) to find that

$$\Delta_{Na} = \frac{\Phi^{*2}}{(1 + \Phi^*)(1 - 2\Phi^*)}. \tag{37}$$

Convert to dimensional units: $\Phi^* = \frac{0.1}{25.8} = 0.00387$

$$\Delta_{Na} = \frac{\Phi^{*2}}{(1 + \Phi^*)(1 - 2\Phi^*)} = 3.0 \times 10^{-5}. \tag{38}$$

This is the fractional charge deflection necessary. Thus, if the initial concentration is 400mM, then the charge deflection concentration is 12μM.

4. Begin by non-dimensionalizing the equation

$$y_{tt} = -\frac{g}{(1+\frac{y}{R})^2} - \frac{gR}{V_T(R+y)}y_t, \tag{39}$$

with initial data

$$y(0) = 0, \qquad y_t(0) = V_0. \tag{40}$$

Rescale time $t = \sigma\tau$ and length $y = LY$ so that

$$LY_{\tau\tau} = -\frac{g\sigma^2}{(1+\frac{LY}{R})^2} - \frac{\sigma g}{V_T(1+\frac{LY}{R})}LY_\tau, \tag{41}$$

with initial data $Y_\tau(0) = \frac{V_0\sigma}{L}$. Pick $L = V_0\sigma$ and $g\sigma^2 = L$ so that $\sigma = \frac{V_0}{g}$, and $L = \frac{V_0^2}{g}$, and $\epsilon = \frac{L}{R} = \frac{V_0^2}{gR}$. The rescaled equation is

$$Y_{\tau\tau} = -\frac{1}{(1+\epsilon Y)^2} - \frac{V_0}{V_T}\frac{1}{(1+\epsilon Y)}Y_\tau, \tag{42}$$

with initial data $Y_\tau(0) = 1$.

Now set $\frac{V_0}{V_T} = \gamma\epsilon$, expand in ϵ, and find that

$$Y(\tau) = -\frac{1}{2}\tau^2 + t + \epsilon(-\frac{1}{12}\tau^4 + (\frac{1}{6}(\gamma+2))\tau^3 - \frac{1}{2}\gamma\tau^2) \tag{43}$$

and the root is

$$\tau = 2 + \epsilon(\frac{4}{3} - \gamma\frac{2}{3}). \tag{44}$$

Convert this back into dimensional parameters to find

$$t = 2\frac{V_0}{g} + \frac{4}{3}\frac{V_0^3}{g^2R} - \frac{2}{3}\frac{V_0^2}{gV_T}. \tag{45}$$

Notice that the correction due to drag *decreases* the time of flight.

2 Matched Asymptotic Expansions

(due Oct. 8, 2015)

1. Find a composite expansion for the solutions of each of the following (include a plot of the approximate solution);

   (a)
   $$\epsilon y'' + y' = a(>0), \qquad y(0) = 0, \qquad y(1) = 1, \tag{46}$$

   (b)
   $$\epsilon y'' + 2y' + y^3 = 0, \qquad y(0) = 0, \qquad y(1) = \frac{1}{2} \tag{47}$$

(c)

$$\epsilon y'' - y' - y^2 = 1, \qquad y(0) = \frac{1}{3}, \qquad y(1) = 1 \tag{48}$$

(d)

$$\epsilon y'' = yy' - y^3, \qquad y(0) = \frac{3}{5}, \qquad y(1) = -\frac{2}{3} \tag{49}$$

(e)

$$\epsilon y'' = 9 - (y')^2, \qquad y(0) = 0, \qquad y(1) = 1 \tag{50}$$

2.1 Solutions

1. (a)

$$y(x) = ax + (1-a)(1 - e^{-\frac{x}{\epsilon}}) + O(\epsilon). \tag{51}$$

(b)

$$y(x) = \frac{1}{\sqrt{3+x}} - \frac{1}{\sqrt{3}} e^{-\frac{2x}{\epsilon}} + O(\epsilon). \tag{52}$$

(c)

$$y(x) = \tan(\tan^{-1}(\frac{1}{3}) - x) + (1 - \tan(\tan^{-1}(\frac{1}{3}) - 1))e^{\frac{x-1}{\epsilon}} \tag{53}$$

(d) There are three solutions

i.

$$y_1(x) = \frac{1}{\frac{5}{3} - x} - \frac{3}{2}\tanh\left(\frac{3}{2}\frac{x-1}{\epsilon} + \tanh^{-1}\frac{4}{9}\right) - \frac{3}{2}. \tag{54}$$

ii.

$$y_2(x) = \frac{4x}{1+2x} - 2\tanh\left(\frac{2x}{\epsilon} - \tanh^{-1}(\frac{3}{10})\right) \tag{55}$$

iii.

$$y_3(x) = H(x - \frac{7}{12})(\frac{12}{13} - \frac{1}{\frac{1}{2} + x}) + H(\frac{7}{12} - x)(\frac{1}{\frac{5}{3} - x} - \frac{12}{13}) - \frac{12}{13}\tanh\left(\frac{12}{13}(\frac{x - \frac{7}{12}}{\epsilon})\right) \tag{56}$$

(e)

$$y(x) = -1 + \epsilon^{\frac{1}{2}} \ln\left(\cosh(\frac{3x-1}{\epsilon^{\frac{1}{2}}})\right) \tag{57}$$

3 QSS Analysis

(due Nov. 12, 2015)

1. The calcium-activated potassium channel has two open states and two closed states (see Fig. 1). The channel has two binding sites for calcium and can open when one of the states is occupied and remains open when the second site is bound. The binding process is a fast process, while the transition between open and closed states is slow. Find the equation describing slow dynamics of the opening of the channel as a function of calcium concentration.

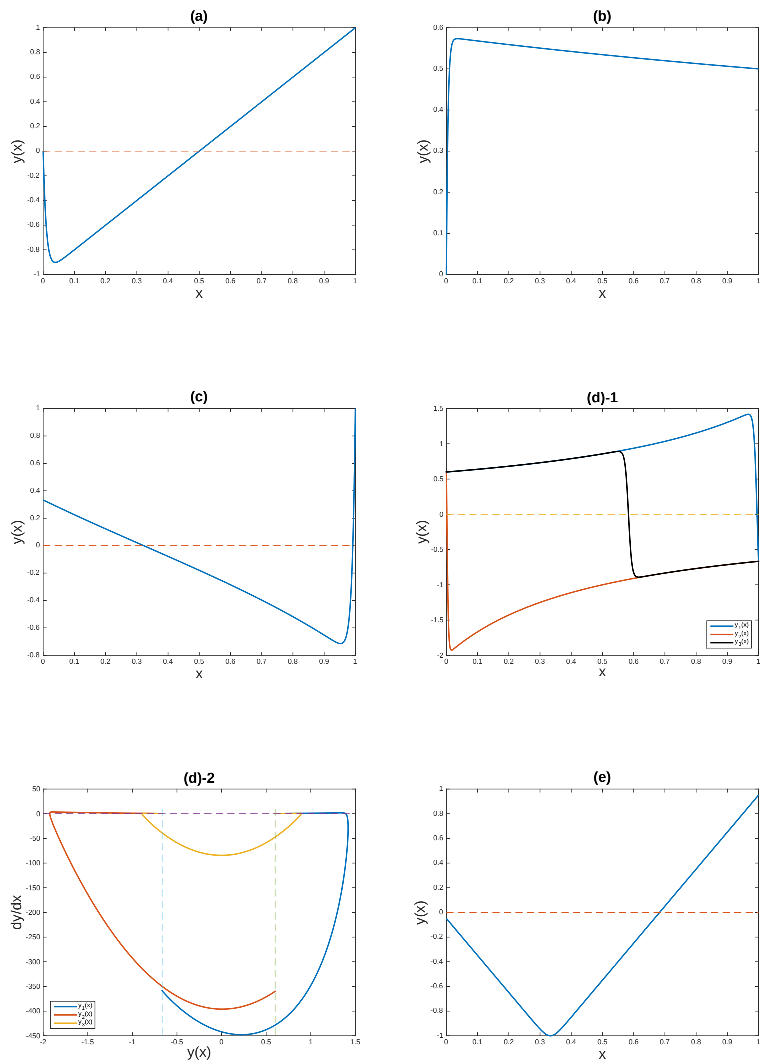
(a)
y(x)
x
(b)
y(x)
x
(c)
y(x)
x
(d)-1
y(x)
x
(d)-2
dy/dx
y(x)
(e)
y(x)
x

Figure 1: Diagram of the four states for the calcium-activated potassium channel.

2. Suppose that an enzyme can bind two substrate molecules, so can exist in one of three states, a free moleule S, a complex with one molecule bound C_1, and a complex with two molecules bound C_2. The corresponding chemical reactions are

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} C_1 \xrightarrow{k_2} P + E \tag{58}$$

$$S + C_1 \underset{k_{-3}}{\overset{k_3}{\rightleftarrows}} C_2 \xrightarrow{k_4} P + C_1 \tag{59}$$

Give a "proper" derivation of the corresponding Michaelis-Menten rate of production of product P, using the assumption that the amount of substrate is much larger than the amount of complex.

3. Consider the process of autocatalytic conversion of a substrate to an enzyme given by

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} C \xrightarrow{k_2} E + E \tag{60}$$

Find the (slow) rate of substrate conversion under the assumption that $k_2 \ll k_{-1}$.

4. Suppose a particle randomly switches between three states

$$S_{-1} \underset{k}{\overset{k}{\rightleftarrows}} S_0 \underset{k}{\overset{k}{\rightleftarrows}} S_1, \tag{61}$$

and in states S_j it moves with velocity jV, $j = -1, 1$, while in state S_0 the particle diffuses with diffusion coefficient D. Suppose the switching rates k are quite large. Find the effective diffusion coefficient for this particle.

3.1 Solutions

1. The differential equation system is

$$\frac{dp_{c_1}}{dt} = -k_1 p_{c_1} + k_{-1} p_{c_2}, \tag{62}$$

$$\frac{dp_{c_2}}{dt} = k_1 p_{c_1} - k_{-1} p_{c_2} - k_2 p_{c_2} + k_{-2} p_{o_1}, \tag{63}$$

$$\frac{dp_{o_1}}{dt} = k_2 p_{c_2} - k_{-2} p_{o_1} - k_3 p_{o_1} + k_{-3} p_{o_2}, \tag{64}$$

$$\frac{dp_{o_2}}{dt} = k_3 p_{o_1} - k_{-3} p_{o_2}, \tag{65}$$

where k_1 and k_3 are calcium dependent, k_1, k_{-1}, k_3, and k_{-3} are large compared to k_2, and k_{-2}. Set $y = p_{c_1} + p_{c_2}$. The qss approximation implies

$$0 = -k_1 p_{c_1} + k_{-1} p_{c_2}, \tag{66}$$

$$0 = k_3 p_{o_1} - k_{-3} p_{o_2}, \tag{67}$$

which implies that

$$p_{c_1} = \frac{k_{-1} y}{k_1 + k_{-1}}, \qquad p_{c_2} = \frac{k_1 y}{k_1 + k_{-1}}, \tag{68}$$

$$p_{o_1} = \frac{k_{-3}}{k_3 + k_{-3}}(1 - y), \qquad p_{o_2} = \frac{k_3}{k_3 + k_{-3}}(1 - y). \tag{69}$$

This implies that

$$\frac{dy}{dt} = -k_2 p_{c_2} + k_{-2} p_{o_1}, \tag{70}$$

$$= -\frac{k_2 k_1}{k_1 + k_{-1}} y + \frac{k_{-2} k_{-3}}{k_3 + k_{-3}}(1 - y), \tag{71}$$

which is the slow equation for the closed-open transition.

2. The differential equation system is

$$\frac{ds}{dt} = -k_1 es + k_{-1} c_1 - k_3 s c_1 + k_{-3} c_2, \tag{72}$$

$$\frac{de}{dt} = -k_1 es + k_{-1} c_1 + k_2 c, \tag{73}$$

$$\frac{dc_1}{dt} = k_1 es - k_{-1} c_1 - k_2 c_1 - k_3 s c_1 + k_{-3} c_2 + k_4 c_2, \tag{74}$$

$$\frac{dc_2}{dt} = k_3 s c_1 - k_{-3} c_2 - k_4 c_2, \tag{75}$$

$$\frac{dp}{dt} = k_2 c_1 + k_4 c_2. \tag{76}$$

Conservation implies that $e + c_1 + c_2 = E_0$ so that we eliminate e and find

$$\frac{ds}{dt} = -k_1 s(E_0 - c_1 - c_2) + k_{-1} c_1 - k_3 s c_1 + k_{-3} c_2, \tag{77}$$

$$\frac{dc_1}{dt} = k_1 s(E_0 - c_1 - c_2) - k_{-1} c_1 - k_2 c_1 - k_3 s c_1 + k_{-3} c_2 + k_4 c_2, \tag{78}$$

$$\frac{dc_2}{dt} = k_3 s c_1 - k_{-3} c_2 - k_4 c_2. \tag{79}$$

Scale the variables using $c_1 = E_0 u_1$, $c_2 = E_0 u_2$, $s = S_0 \sigma$ and $t = \frac{S_0}{k_{-1} E_0} \tau$ and find

$$\frac{d\sigma}{d\tau} = -\kappa_1 \sigma(1 - u_1 - u_2) + u_1 - \kappa_3 \sigma u_1 + \kappa_{-3} u_2, \tag{80}$$

$$\epsilon \frac{du_1}{d\tau} = \kappa_1 \sigma(1 - u_1 - u_2) - u_1 - \kappa_2 u_1 - \kappa_3 \sigma u_1 + \kappa_{-3} u_2 + \kappa_4 u_2, \tag{81}$$

$$\epsilon \frac{du_2}{d\tau} = \kappa_3 \sigma u_1 - \kappa_{-3} u_2 - \kappa_4 u_2, \tag{82}$$

where $\kappa_1 = \frac{S_0 k_1}{k_{-1}}$, $\kappa_2 = \frac{k_2}{k_{-1}}$, $\kappa_3 = \frac{S_0 k_3}{k_{-1}}$, , $\kappa_{-3} = \frac{k_{-3}}{k_{-1}}$,$\kappa_4 = \frac{k_4}{k_{-1}}$, and $\epsilon = \frac{E_0}{S_0}$. The quasi-steady approximation takes

$$0 = \kappa_1\sigma(1-u_1-u_2) - u_1 - \kappa_2 u_1 - \kappa_3\sigma u_1 + \kappa_{-3}u_2 + \kappa_4 u_2, \tag{83}$$

$$0 = \kappa_3\sigma u_1 - \kappa_{-3}u_2 - \kappa_4 u_2 \tag{84}$$

which we solve to find

$$u_1 = \frac{(\kappa_{-3}+\kappa_4)\kappa_1\sigma}{\kappa_1\kappa_3\sigma^2 + \kappa_1\sigma(\kappa_{-3}+\kappa_4) + (\kappa_2+1)(\kappa_4+\kappa_{-3})}, \tag{85}$$

$$u_2 = \frac{\kappa_1\kappa_3\sigma^2}{\kappa_1\kappa_3\sigma^2 + \kappa_1\sigma(\kappa_{-3}+\kappa_4) + (\kappa_2+1)(\kappa_4+\kappa_{-3})}. \tag{86}$$

It follows that

$$\frac{dp}{dt} = E_0\kappa_1\frac{k_2(\kappa_{-3}+\kappa_4)\sigma + k_4\kappa_3\sigma^2}{\kappa_1\kappa_3\sigma^2 + \kappa_1\sigma(\kappa_{-3}+\kappa_4) + (\kappa_2+1)(\kappa_4+\kappa_{-3})}. \tag{87}$$

3. The differential equation system is

$$\frac{ds}{dt} = -k_1 es + k_{-1}c, \tag{88}$$

$$\frac{de}{dt} = -k_1 es + k_{-1}c + k_2 c, \tag{89}$$

$$\frac{dc}{dt} = k_1 es - k_{-1}c - k_2 c. \tag{90}$$

Set $e = E_0 - c$, so that

$$\frac{ds}{dt} = -k_1 s(E_0 - c) + k_{-1}c, \tag{91}$$

$$\frac{dc}{dt} = k_1 s(E_0 - c) - k_{-1}c - k_2 c. \tag{92}$$

Under the assumption that $k_2 \ll k_{-1}$, we take s to be in quasi-equilibrium so that

$$c = \frac{k_1 E_0 s}{k_1 s + k_{-1}}, \tag{93}$$

and then the slow evolution is

$$\frac{d(s+c)}{dt} = -k_2 c, \tag{94}$$

or

$$\frac{d}{dt}\left(s + \frac{E_0 s}{s+K_d}\right) = -\frac{k_2 E_0 s}{s+K_d}, \tag{95}$$

where $K_d = \frac{k_{-1}}{k_1}$, which is a differential equation for the (slow) conversion of substrate to enzyme.

4. The deterministic part of the motion is

$$\frac{dy}{dt} = Vn \tag{96}$$

where n has three states $n = -1, 0, 1$ with transitions

$$S_{-1} \underset{k}{\overset{k}{\rightleftarrows}} S_0 \underset{k}{\overset{k}{\rightleftarrows}} S_1 \tag{97}$$

The master equations are

$$\begin{aligned} \frac{\partial p_{-1}}{\partial t} &= kp_0 - kp_{-1} + V\frac{\partial p_{-1}}{\partial x}, &(98)\\ \frac{\partial p_0}{\partial t} &= kp_{-1} + kp_1 - 2kp_0 + D\frac{\partial^2 p_0}{\partial x^2}, &(99)\\ \frac{\partial p_1}{\partial t} &= kp_0 - kp_1 - V\frac{\partial p_1}{\partial x}. &(100)\end{aligned}$$

We assume that k is large so introduce the parameter $\epsilon = \frac{1}{k}$. Then, the equations become

$$\begin{aligned} \frac{\partial p_{-1}}{\partial t} &= \frac{1}{\epsilon}(p_0 - p_{-1}) + V\frac{\partial p_{-1}}{\partial x}, &(101)\\ \frac{\partial p_0}{\partial t} &= \frac{1}{\epsilon}(p_{-1} + p_1 - 2p_0) + D\frac{\partial^2 p_0}{\partial x^2}, &(102)\\ \frac{\partial p_1}{\partial t} &= \frac{1}{\epsilon}(p_0 - p_1) - V\frac{\partial p_1}{\partial x}. &(103)\end{aligned}$$

The nullspace of the matrix is spanned by the vector

$$\phi = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{104}$$

We set

$$p = \begin{pmatrix} p_{-1} \\ p_0 \\ p_1 \end{pmatrix} = v\phi + \begin{pmatrix} w_{-1} \\ w_0 \\ w_1 \end{pmatrix}, \tag{105}$$

and find the equation for v by projecting

$$\begin{aligned} \frac{\partial v}{\partial t} &= V\frac{\partial p_{-1}}{\partial x} - V\frac{\partial p_1}{\partial x} + D\frac{\partial^2 p_0}{\partial x^2} &(106)\\ &= V\frac{\partial}{\partial x}(w_{-1} - w_1) + \frac{1}{3}D\frac{\partial^2}{\partial x^2}(v + w_0). &(107)\end{aligned}$$

The vector w must satisfy the equation

$$\frac{1}{\epsilon}\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} w = \frac{1}{3}\begin{pmatrix} -V\frac{\partial v}{\partial x} \\ 0 \\ V\frac{\partial v}{\partial x} \end{pmatrix}. \tag{108}$$

The solution of this problem has

$$w_{-1} - w_1 = \frac{2\epsilon}{3} V \frac{\partial v}{\partial x}. \tag{109}$$

Thus, the effective Fokker-Planck equation (ignoring higher order derivatives) is

$$\frac{\partial v}{\partial t} = D_{eff} \frac{\partial^2 v}{\partial x^2} \tag{110}$$

where

$$D_{eff} = \frac{1}{3} D + \frac{2\epsilon}{3} V^2, \tag{111}$$

which, in case you are checking, has the correct units.

4 Multiscale/Averaging

(due Dec. 3, 2015)

1. Using your choice of method (multiscale analysis or averaging), find approximate solutions for the equations

   (a)
   $$u'' + \epsilon (u')^3 + u = 0 \tag{112}$$

   (b)
   $$u'' - \epsilon \cos t (u')^2 + u = 0 \tag{113}$$

   (c)
   $$u'' + \epsilon \cos(\epsilon t)(u')^3 + u = 0 \tag{114}$$

   (d)
   $$u'' + \epsilon(\epsilon + u')u' + u = 0 \tag{115}$$

   (e)
   $$u'' + \epsilon(\epsilon + (u')^2)u' + u = 0 \tag{116}$$

   Remark: For this problem, use higher order averaging. Why is this method preferable for this problem?

4.1 Solutions

1. (a) Using multiscaling, the equation is rewritten as

$$\frac{\partial^2 u}{\partial \tau^2} + 2\epsilon \frac{\partial^2 u}{\partial \tau \partial s} + \epsilon^2 \frac{\partial^2 u}{\partial s^2} + \epsilon (\frac{\partial u}{\partial \tau} + \epsilon \frac{\partial^u}{\partial s})^3 + u = 0 \tag{117}$$

   Setting

$$u = A(s)(e^{i\tau + i\phi(s)} + e^{-i\tau - i\phi(s)} + \epsilon u_1(\tau, s) + O(\epsilon^2), \tag{118}$$

we require

$$\frac{\partial^2 u_1}{\partial \tau^2} + u_1 + 2\frac{\partial}{\partial s}(A(s)(ie^{i\tau+i\phi(s)} - ie^{-i\tau-i\phi(s)}) + A^3(s)(ie^{i\tau+i\phi(s)} - ie^{-i\tau-i\phi(s)})^3 = 0. \tag{119}$$

The solvability condition is

$$2A'i - 2A\phi' + A^3 i = 0 \tag{120}$$

which, after separating into real and imaginery parts, gives

$$2A' + A^3 = 0, \qquad \phi' = 0 \tag{121}$$

Notice that this is *not* structurally stable but that is the nature of the problem and nothing can be done to rectify this.

(b) Use multiscaling to write the equation as

$$\frac{\partial^2 u}{\partial \tau^2} + 2\epsilon\frac{\partial^2 u}{\partial \tau \partial s} + \epsilon^2\frac{\partial^2 u}{\partial s^2} + \epsilon\cos(\tau)(\frac{\partial u}{\partial \tau} + \epsilon\frac{\partial u}{\partial s})^2 + u = 0 \tag{122}$$

where $\tau = t$, $\sigma = \epsilon t$. With $u = A(\sigma)\cos(\tau + \phi(\sigma)) + \epsilon u_1$ we find at $O(\epsilon)$:

$$\frac{\partial^2 u_1}{\partial \tau^2} + u_1 - 2\frac{dA}{ds}\sin(\tau+\phi) - 2A\frac{d\phi}{ds}\cos(\tau+\phi) + A^2\cos(\tau)\cos^2(\tau+\phi) = 0 \tag{123}$$

Thus, we require

$$\frac{dA}{ds} - \frac{1}{8}A^2\sin\phi = 0 \tag{124}$$

and

$$\frac{d\phi}{dt} - \frac{3}{8}A\cos\phi = 0 \tag{125}$$

One cannot find

(c) Use multiscaling to write the equation as

$$\frac{\partial^2 u}{\partial \tau^2} + 2\epsilon\frac{\partial^2 u}{\partial \tau \partial s} + \epsilon^2\frac{\partial^2 u}{\partial s^2} + \epsilon\cos(s)(\frac{\partial u}{\partial \tau} + \epsilon\frac{\partial^u}{\partial s})^3 + u = 0 \tag{126}$$

Set $u = A(s)\cos(t + \phi(s) + \epsilon u_1$ and find at $O(\epsilon)$:

$$\frac{\partial^2 u_1}{\partial \tau^2} + u_1 - 2\frac{dA}{ds}\sin(\tau+\phi) - 2A\frac{d\phi}{ds}\cos(\tau+\phi) + A^3\cos(s)\sin^3(\tau+\phi) = 0 \tag{127}$$

and then require

$$\frac{dA}{ds} = \frac{3}{8}A^3\cos(s), \qquad \frac{d\phi}{ds} = 0 \tag{128}$$

(d) For this problem, one can use either averaging or multiscaling, but in either case, one must go beyond the first order to get a correct approximation. Using a multiscale approach, choose a slow time scale $s = \epsilon^2 t$ and write the equation as

$$\frac{\partial^2 u}{\partial \tau^2} + 2\epsilon^2\frac{\partial^2 u}{\partial \tau \partial s} + \epsilon^4\frac{\partial^2 u}{\partial s^2} + \epsilon(\epsilon + \frac{\partial u}{\partial \tau} + \epsilon^2\frac{\partial u}{\partial s})(\frac{\partial u}{\partial \tau} + \epsilon^2\frac{\partial u}{\partial s}) + u = 0 \tag{129}$$

and then the approximation $u = u_0 + \epsilon u_1 + \epsilon^2 u_2$ yields

$$\frac{\partial^2 u_1}{\partial \tau^2} + u_1 + (\frac{\partial u_0}{\partial \tau})^2 = 0 \tag{130}$$

and

$$\frac{\partial^2 u_2}{\partial \tau^2} + u_2 + 2\frac{\partial^2 u_0}{\partial \tau \partial s} + \frac{\partial u_0}{\partial \tau} + 2\frac{\partial u_0}{\partial \tau}\frac{\partial u_1}{\partial \tau} = 0 \tag{131}$$

Take $u_0 = A(s)\cos(t + \phi(s))$ and then $u_1 = A^2(\frac{1}{6}\cos(2\tau + 2\phi) - \frac{1}{2})$, and the equation for u_2 becomes

$$\frac{\partial^2 u_2}{\partial \tau^2}+u_2-2\frac{dA}{ds}\sin(\tau+\phi)-2A\frac{d\phi}{ds}\cos(t+\phi)+A\sin(\tau+\phi)+\frac{1}{3}A^3\sin(\tau+\phi)\sin(2\tau+2\phi) = 0 \tag{132}$$

(e) To use averaging, first introduce polar coordinates $u = R\cos\theta$, $u' = R\sin\theta$. Then, require

$$R'\cos\theta - R\theta' sin\theta - R\sin\theta = 0 \tag{133}$$

and

$$R'\sin\theta + R\theta'\cos\theta + \epsilon(\epsilon + (R\sin\theta)^2)R\sin\theta + R\cos\theta = 0 \tag{134}$$

Combining these the appropriate way, we find

$$R' + \epsilon(\epsilon + (R\sin\theta)^2)R\sin^2\theta = 0 \tag{135}$$

and

$$\theta' + 1 + \epsilon(\epsilon + (R\sin\theta)^2)\sin\theta\cos\theta = 0 \tag{136}$$

Finally, set $\theta = \psi - t$ so that

$$R' = -\epsilon(\epsilon + R^2\sin^2(\psi - t))R\sin^2(\psi - t) \tag{137}$$
$$\psi' = -\epsilon(\epsilon + R^2\sin^2(\psi - t))\sin(\psi - t)\cos(\psi - t) \tag{138}$$

Now we seek a change of variables

$$R = \rho + \epsilon R_1(\rho, \phi, t) + \epsilon^2 R_2(\rho, \phi, t)\cdots, \tag{139}$$
$$\psi = \phi + \epsilon P_1(\rho, \phi, t) + \epsilon^2 P_2(\rho, \phi, t) + \cdots \tag{140}$$

so that the equations have the form

$$\rho' = \epsilon H_0(\rho, \phi) + \epsilon^2 H_1(\rho, \phi) + \cdots \tag{141}$$
$$\phi' = \epsilon G_0(\rho, \phi) + \epsilon^2 G(\rho, \phi) \tag{142}$$

This means

$$O(\epsilon) : H_0 + R_{1_t} = -\rho^3\sin^4(\psi - t), \qquad G_0 + P_{1_t} = -\rho^3\sin^3(\phi - t)\cos(\phi - t) \tag{143}$$

so that

$$H_0 = -\frac{3}{8}\rho^3, \qquad R_1 = -\frac{1}{4}\sin(\phi - t)\cos^3(\phi - t)\rho^3 + \frac{5}{8}\rho^3\cos(\phi - t)\sin(\phi - t) \tag{144}$$

and

$$G_0 = 0, \qquad P_1 = \frac{1}{4}\rho^3 \sin^4(\phi - t) \tag{145}$$

To next order the equations are complicated, but the answer is

$$H_1 = -\frac{1}{2}\rho, \qquad G_1 = \frac{5}{128}\rho^6 + \frac{3}{512}\rho^5 \tag{146}$$

This gives us the averaged equations we want, namely

$$\rho' = -\epsilon\frac{3}{8}\rho^3 - \epsilon^2\frac{1}{2}\rho. \tag{147}$$

The solution of this equation is

$$\rho(t) = \sqrt{\frac{4\epsilon}{A\exp(\epsilon^2 t) - 3}} \tag{148}$$

where $A = 3 + \frac{4\epsilon}{\rho^2(0)} > 3$, which behaves much differently than the leading order solution, which decays algebraically.

5 Homogenization

(due Dec. 17, 2013)

1. A laminate material consists of many thin layers of alternating materials with different thermal conductivities. Suppose the layers have thermal conductivities D_i and thicknesses l_i, $i = 1, 2$. Find the effective conductivity tensor for the composite material.

2. Find the effective diffusion tensor for a periodic medium in which the diffusion coefficient is rapidly varying in space, according to

$$D(x, y) = \exp\left(\alpha(\frac{x}{\epsilon}) + \beta(\frac{y}{\epsilon})\right), \tag{149}$$

where $\alpha(x)$ and $\beta(y)$ are periodic functions with period a and b, respectively.

3. Find the coupled phase equations for a system of coupled lambda-omega ($\lambda - \omega$) systems, of the form

$$\frac{d}{dt}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \lambda(r_i) & -\omega \\ \omega & \lambda(r_i) \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} + \epsilon\begin{pmatrix} \sum_j g_{ij} \\ 0 \end{pmatrix}, \tag{150}$$

$i = 1, \cdots, N$, where $r_i = x_i^2 + y_i^2$, $\epsilon \ll 1$, $\lambda(1) = 0$, $\lambda'(1) < 0$, and

(a)

$$g_{ij} = c_{ij}(x_j - x_i), \tag{151}$$

(diffusive coupling)

(b)

$$g_{ij} = c_{ij}x_j \tag{152}$$

(synaptic coupling).

5.1 Solutions

1. Use the answer to Problem 2 to calculate the effective diffusion tensor. We find, in the direction normal to the layers, the diffusion coefficient is

$$D_{eff} = \frac{D_1 D_2 (l_1 + l_2)}{l_1 D_2 + l_2 D_1}. \tag{153}$$

In the direction parallel to the lamination, the diffusion coefficient is

$$D_{eff} = \frac{l_1 D_1 + l_2 D_2}{l_1 + l_2}. \tag{154}$$

2. Follow standard arguments to conclude that

$$D_{eff} = \frac{1}{V_\Omega} \int_\Omega D(I + \nabla W) dV, \tag{155}$$

where W is the vector that satisfies

$$\nabla \cdot (D \nabla W) = -\nabla \cdot (DI), \tag{156}$$

where I is the identity matrix.

For this problem with $D = \exp(\alpha(x) + \beta(y))$, W must satisfy

$$\nabla \cdot (\exp(\alpha(x) + \beta(y)) \nabla W) = -\nabla \cdot (\exp(\alpha(x) + \beta(y)) I), \tag{157}$$

or

$$(\exp(\alpha(x)) w^1_x)_x = -(\exp(\alpha(x))_x, \qquad (\exp(\beta(y)) w^2_y)_y = -(\exp(\beta(y))_y, \tag{158}$$

which integrates once to

$$\exp(\alpha(x)) w^1_x = K_1 - \exp(\alpha(x)), \qquad \exp(\beta(y)) w^2_y = K_2 - \exp(\beta(y), \tag{159}$$

$$w^1_x = K_1 \exp(-\alpha(x)) - 1, \qquad w^2_y = K_2 \exp(-\beta(y)) - 1, \tag{160}$$

where

$$K_1 = \left(\frac{1}{a} \int_0^a \exp(-\alpha(x)) dx\right)^{-1}, \qquad K_2 = \left(\frac{1}{b} \int_0^b \exp(-\beta(y)) dy\right)^{-1} \tag{161}$$

It follows from (155) that

$$D_{eff} = \begin{pmatrix} K_1 \frac{1}{b} \int_0^b \exp(\beta(y)) dy & 0 \\ 0 & K_2 \frac{1}{a} \int_0^a \exp(\alpha(x)) dx \end{pmatrix}. \tag{162}$$

3. The multiscaled equations are

$$\frac{d}{dt}\begin{pmatrix} x_i \\ y_i \end{pmatrix} + \epsilon\frac{d}{d\sigma}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \lambda(r_i) & -\omega \\ \omega & \lambda(r_i) \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} + \epsilon\begin{pmatrix} \sum_j g_{ij} \\ 0 \end{pmatrix}, \tag{163}$$

so setting $x_i = x_i^0 + \epsilon x_i^1 + \cdots$, $y_i = y_i^0 + \epsilon y_i^1 + \cdots$, we find

$$\frac{d}{dt}\begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} = \begin{pmatrix} \lambda(r_i^0) & -\omega \\ \omega & \lambda(r_i^0) \end{pmatrix}\begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} \tag{164}$$

and

$$\begin{aligned} L\begin{pmatrix} x_i^1 \\ y_i^1 \end{pmatrix} &\equiv \frac{d}{dt}\begin{pmatrix} x_i^1 \\ y_i^1 \end{pmatrix} - \begin{pmatrix} \lambda'(r_0)\frac{(x_i^0)^2}{r_0} & -\omega + \lambda'(r_0)\frac{x_i^0 y_i^0}{r_0} \\ \omega + \lambda'(r_0)\frac{x_i^0 y_i^0}{r_0} & \lambda'(r_0)\frac{(y_i^0)^2}{r_0} \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} && (165) \\ &= -\frac{d}{d\sigma}\begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} + \begin{pmatrix} \sum_j g_{ij}^0 \\ 0 \end{pmatrix}, && (166) \end{aligned}$$

The leading order periodic solution is given by

$$\begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} = \begin{pmatrix} \cos(\omega t + \phi_i) \\ \sin(\omega t + \phi_i) \end{pmatrix}. \tag{167}$$

It is easy to see that the linearized operator $L\begin{pmatrix} x \\ y \end{pmatrix}$ has a nullspace

$$\begin{pmatrix} -\sin(\omega t + \phi_i) \\ \cos(\omega t + \phi_i) \end{pmatrix} \tag{168}$$

The adjoint operator is

$$L^*\begin{pmatrix} u \\ v \end{pmatrix} = -\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \lambda'(r_0)\frac{(x_i^0)^2}{r_0} & \omega + \lambda'(r_0)\frac{x_i^0 y_i^0}{r_0} \\ -\omega + \lambda'(r_0)\frac{x_i^0 y_i^0}{r_0} & \lambda'(r_0)\frac{(y_i^0)^2}{r_0} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{169}$$

and one can directly verify that the nullspace is spanned by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\sin(\omega t + \phi_i) \\ \cos(\omega t + \phi_i) \end{pmatrix}. \tag{170}$$

It follows that the solvability condition for the order ϵ system is

$$\frac{d\phi_i}{d\sigma} = -\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin(\omega t + \phi_i)\sum_j g_{ij}^0. \tag{171}$$

We calculate

$$\begin{aligned} \int_0^{\frac{2\pi}{\omega}} \sin(\omega t + \phi_i) g_{ij}^0 dt &= \int_0^{\frac{2\pi}{\omega}} \sin(\omega t + \phi_i) g_{ij}(x_j^0 - \alpha x_i^0) dt && (172) \\ &= \int_0^{\frac{2\pi}{\omega}} \sin(\omega t + \phi_i) c_{ij}(\cos(\omega t + \phi_j) - \alpha\cos(\omega t + \phi_i)) dt && (173) \\ &= \frac{1}{\omega}\int_0^{2\pi} \sin(\tau) c_{ij}(\cos(\tau + \phi_j - \phi_i) - \alpha\cos(t) dt && (174) \\ &= -\frac{\pi}{\omega}\sin(\phi_j - \phi_i) && (175) \end{aligned}$$

It follows that the phase equations are

$$\frac{d\phi_i}{d\sigma} = \frac{1}{2} \sum_j c_{ij} \sin(\phi_j - \phi_i). \tag{176}$$