Math 6420 Partial Differential Equations

1 Introduction

Partial Differential equations is a vast topic with a long history. Here is a list of some of the most important PDE's:

1. Laplace's Equation
$$\nabla^2 u = 0 \tag{1}$$

2. Poisson Equation
$$\nabla^2 u = f \tag{2}$$

3. Stokes Equation
$$\nabla^2 \mathbf{u} = \nabla p, \qquad \nabla \cdot \mathbf{u} = 0 \tag{3}$$

4. Diffusion Equation
$$\frac{\partial u}{\partial t} = \nabla^2 u \tag{4}$$

5. Fisher's Equation
$$\frac{\partial u}{\partial t} = \nabla^2 u + u(1-u) \tag{5}$$

6. Bistable Equation
$$\frac{\partial u}{\partial t} = \nabla^2 u + u(1-u)(u-\alpha) \tag{6}$$

7. Fokker-Planck Equation
$$\frac{\partial p}{\partial t} = \frac{\partial^2}{\partial x^2}(Dp) + \frac{\partial}{\partial x}(V'(x)p) \tag{7}$$

8. Black-Scholes Equation
$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0 \tag{8}$$

9. Motion by Mean Curvature Equation
$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) \tag{9}$$

10. Allen-Cahn Equation
$$\frac{\partial u}{\partial t} = \nabla^2 u + f'(u) \tag{10}$$

11. Porous Medium Equation
$$\frac{\partial u}{\partial t} = \nabla^2 (u^m), \qquad m > 1 \tag{11}$$

12. Cahn-Hilliard Equation

$$\frac{\partial u}{\partial t} = -\nabla^2(\nabla^2 u + f'(u)) \tag{12}$$

13. Ginzberg-Landau Equation

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda(r) & \omega \\ -\omega & \lambda(r) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + D\frac{\partial^2}{\partial x^2}\begin{pmatrix} u \\ v \end{pmatrix} \tag{13}$$

where $r^2 = u^2 + v^2$.

14. Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \tag{14}$$

15. Helmholtz Equation

$$\nabla^2 u - ku = 0 \tag{15}$$

16. Telegraphers Equation

$$\frac{\partial^2 u}{\partial t^2} + \mu\frac{\partial u}{\partial t} = \nabla^2 u \tag{16}$$

17. Beam Equation

$$\frac{\partial^2 u}{\partial t^2} = -\nabla^2(\nabla^2 u) \tag{17}$$

18. Biharmonic Equation

$$\nabla^2(\nabla^2 u) = 0 \tag{18}$$

19. Schrodinger equation

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \tag{19}$$

20. Inviscid Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{20}$$

21. Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2} \tag{21}$$

22. Nonlinear Schrodinger equation

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + k|\psi|^2\psi \tag{22}$$

23. Korteweg-deVries Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3} \tag{23}$$

24. Klein-Gordon Equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u - ku \tag{24}$$

25. Sine-Gordon Equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u - \sin(u) \tag{25}$$

26. Eikonal Equation

$$|\nabla u| = c(x) \tag{26}$$

27. Navier-Stokes Equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mu \nabla^2 \mathbf{u} - \nabla p, \qquad \nabla \cdot u = 0 \tag{27}$$

28. Navier's equations of linear Elasticity

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + (\mu + \lambda)\nabla(\nabla \cdot \mathbf{u}) \tag{28}$$

29. Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{29}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{30}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{31}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{32}$$

30. Einstein's Equations

2 Course Outline

1. Scalar conservation laws and first-order equations
   (a) Linear transport equation
   (b) Traffic dynamics
   (c) Weak solutions and shock waves
   (d) Method of characteristics for quasilinear equations
   (e) General
   first-order equations
2. Waves and Vibrations
   (a) Introduction: waves on a string

(b) One-dimensional wave equation
(c) The D'Alembert formula and characteristics
(d) Classification of second-order linear equations
(e) Multi-dimensional wave equation

3. Diffusion
   (a) The one-dimensional diffusion equation
   (b) Uniqueness and maximum principles
   (c) Fundamental solution and the global Cauchy problem
   (d) Random walks
   (e) Some nonlinear problems: traveling waves
4. The Laplace equation
   (a) Harmonic functions, mean value theorems, and maximum principles
   (b) Fundamental solution and the global Cauchy problem
   (c) Green's functions
   (d) Potential theory
5. Variational Problems
   (a) Linear operators and duality
   (b) Lax-Milgram theorem and minimization of bilinear forms
   (c) Galerkin method
   (d) Variational formulation of Poissons equation in 1D

Recommended texts:

- S Salsa, Partial differential equations in action (Springer 2009)
- J Ockendon, S Howison, A Lacey and A Movchan, Applied partial differential equations, 2nd ed. (Oxford University Press 2003)