Math 6420
Homework Exercises

1 First Order PDE's (due Feb. 17, 2016)

1. Find the solution of the problem

$$u_t + cu_x = f(x,t), \qquad u(x,0) = 0, \tag{1}$$

where $f(x,t) = \exp(-t)\sin(x)$.

2. Solve the Burgers equation $u_t + uu_x = 0$ with initial data

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}. \tag{2}$$

3. A reasonable model for automobile speed in a long single lane tunnel is

$$v(u) = v_m \cdot \begin{cases} 1 & 0 \le u \le u_c, \\ \frac{\ln(\frac{u_m}{u})}{\ln(\frac{u_m}{u_c})} & u_c \le u \le u_m, \end{cases} \tag{3}$$

where u is the density of cars. Typical parameter values are $u_c = 7$car/km, $v_m =$ 90km/h, $u_m = 110$car/km, $\ln(\frac{u_m}{u_c} = 2.75$.

Suppose the initial density is

$$u = \begin{cases} u_m & x < 0 \\ 0 & x > 0 \end{cases}. \tag{4}$$

Determine the trajectory in the x,t plane of a car starting at position $x = x_0 < 0$ and determine the time it takes for the car to enter the tunnel, and for it to pass through a tunnel of length 5 km.

4. Determine the solvability conditions for the linear problem

$$a(x,y)u_x - u_y = -u, \qquad u(x,x^2) = g(x). \tag{5}$$

Examine the specific case $a(x,y) = \frac{y}{2}$, $g(x) = \exp(-\gamma x^2)$.

5. Solve the Cauchy problem

$$u_x^2 + u_y^2 = 4u, \qquad u(x,0) = x^2 \tag{6}$$

2 The Wave Equation (due March 21, 2016)

1. Solve the problem
$$u_{tt} = u_{xx}, \qquad 0 < x < 1, \qquad t > 0, \tag{7}$$
subject to initial conditions $u(x,0) = u_t(x,0) = 0$ and boundary conditions $u_x(0,t) = 1$, $u(1,t) = 0$.

2. Find the characteristics for the equation $u_{tt} = tu_{xx}$.

3. Is the problem
$$u_{tt} = u_{xx}, \qquad -t < x < t, \tag{8}$$
subject to conditions $u(x,x) = f(x)$, $u(x,-x) = g(x)$ with $f(0) = g(0)$ well posed?

4. Find characteristics and use these to find the general solution of $t^2u_{tt}+2tu_{xt}+u_{xx}-u = 0$.

5. Solve the problem $u_{tt} - c^2\nabla^2 u = 0$ in two and three dimensional space for $t > 0$ subject to initial conditions $u(\mathbf{x},0) = 0, u_t(\mathbf{x},0) = h(|\mathbf{x}|)$, where $h(r) = H(1-r)$ for $r > 0$, where H is the heaviside function. Plot the solution $u(0,t)$.

3 The Diffusion Equation (due April 11, 2016)

1. Suppose that particles in discrete boxes of size Δx leave box j to box $j \pm 1$ at the rate $\frac{\lambda_{j\pm1}}{\Delta x^2}$, where $\lambda_j = \lambda(j\Delta x)$ for some smooth function $\lambda(x)$. Derive the limiting diffusion equation, written in conservation form. Identify the different flux terms.

2. A simple model for the dispersal of seeds is that once they become airborne, they diffuse and advect with the wind and drop onto the ground at a linear rate. Thus, the density of seeds in the air (in a one dimensional region) is specified by u where
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} - ku, \tag{9}$$
and the amount of seed on the ground is determined by q where
$$\frac{\partial q}{\partial t} = ku, \tag{10}$$
starting from initial data $u(x,0) = \delta(x)$, $q(x,0) = 0$.

   Find $q(x,\infty) = \lim_{t\to\infty} q(x,t)$, and plot it for $v = 0$ and $v > 0$, using nondimensional variables.

3. (a) Find solutions of the diffusion equation $u_t = u_{xx}$ of the form $u(x,t) = U(\frac{x}{\sqrt{t}})$ expressed in terms of the error function
$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x^2)dx. \tag{11}$$

(b) Use this solution to solve the equation $u_t = Du_{xx}$ on the domain $x > 0$, $t > 0$, subject to conditions $u(0,t) = 1$ and $u(x,0) = 0$ for $x > 0$.

(c) Find and plot the curve $x = X(t)$ along which $u(X(t),t) = \frac{1}{2}$.

(d) Calculate the total amount of u for $x > 0$, $\int_0^\infty u(x,t)dx$, as a function of t.

4. Suppose the population of some organism is governed by the equation

$$u_t = Du_{xx} + ku(1-u) \tag{12}$$

on the interval $0 < x < L$ subject to boundary conditions $u_x = 0$ at $x = 0$, and $Du_x + \alpha u = 0$ at $x = L$. Under what conditions on the parameters D, L, k, and α can such a population survive?

4 Laplace's Equation (due April 28, 2016)

1. Find the solution $u(r,\theta)$ of Laplace's equation $\nabla^2 u = 0$ on the interior of a circular domain of radius R subject to Dirichlet boundary data $u(R,\theta) = \cos(n\theta)$ for any integer n.

2. Find the solution $u(r,\theta)$ of Laplace's equation $\nabla^2 u = 0$ on the exterior of a circular domain of radius R subject to Dirichlet boundary data $u(R,\theta) = \cos(n\theta)$ for any integer n.

3. Under what conditions on λ does a solution $u(r,\theta)$ of

$$\nabla^2 u = -1, \tag{13}$$

on the annulus $1 < r < 2$, with boundary conditions $u_\nu = \cos\theta$ at $r = 1$, and $u_\nu = \lambda \cos^2\theta$ at $r = 2$, where u_ν refers to the outward normal derivative? Find a solution when it exists. Is it unique?

Hint: Make use of Green's integral identity $\int_\Omega (v\nabla^2 u - u\nabla^2 v)dx = \int_{\partial\Omega}(vu_\nu - uv_\nu)d\sigma$ with $v = 1$.

4. Find the Green's function for a two dimensional disc of radius R.

5. Find the Green's function for the unit hemisphere in three dimensions $x^2+y^2+z^2 \leq 1$, $z > 0$.