1 First Order PDE’s (due Feb. 17, 2016)

1. Find the solution of the problem

\[ u_t + cu_x = f(x,t), \quad u(x,0) = 0, \quad (1) \]

where \( f(x,t) = \exp(-t) \sin(x) \).

2. Solve the Burgers equation \( u_t + uu_x = 0 \) with initial data

\[
\begin{cases} 
1 & x \leq 0 \\
1 - x & 0 < x < 1 \\
0 & x \geq 1 
\end{cases}
\]

3. A reasonable model for automobile speed in a long single lane tunnel is

\[
v(u) = v_m \cdot \begin{cases} 
1 & 0 \leq u \leq u_c, \\
\ln\left(\frac{u_m}{u_x}\right) & u_c \leq u \leq u_m, 
\end{cases}
\]

where \( u \) is the density of cars. Typical parameter values are \( u_c = 7 \) car/km, \( v_m = 90 \) km/h, \( u_m = 110 \) car/km, \( \ln\left(\frac{u_m}{u_x}\right) = 2.75 \).

Suppose the initial density is

\[
u = \begin{cases} 
u_m & x < 0 \\
0 & x > 0 
\end{cases}
\]

Determine the trajectory in the \( x, t \) plane of a car starting at position \( x = x_0 < 0 \) and determine the time it takes for the car to enter the tunnel, and for it to pass through a tunnel of length 5 km.

4. Determine the solvability conditions for the linear problem

\[
a(x,y)u_x - u_y = -u, \quad u(x,x^2) = g(x). \quad (5)\]

Examine the specific case \( a(x,y) = \frac{\gamma}{2}, \ g(x) = \exp(-\gamma x^2) \).

5. Solve the Cauchy problem

\[
u_x^2 + u_y^2 = 4u, \quad u(x,0) = x^2 \quad (6)\]
2 The Wave Equation (due March 21, 2016)

1. Solve the problem

\[ u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0, \]

subject to initial conditions \( u(x,0) = u_t(x,0) = 0 \) and boundary conditions \( u_x(0,t) = 1, \ u(1,t) = 0 \).

2. Find the characteristics for the equation \( u_{tt} = tu_{xx} \).

3. Is the problem

\[ u_{tt} = u_{xx}, \quad -t < x < t, \]

subject to conditions \( u(x,x) = f(x), \ u(x,-x) = g(x) \) with \( f(0) = g(0) \) well posed?

4. Find characteristics and use these to find the general solution of \( t^2u_{tt} + 2tu_{xt} + u_{xx} - u = 0 \).

5. Solve the problem \( u_{tt} - c^2\nabla^2 u = 0 \) in two and three dimensional space for \( t > 0 \) subject to initial conditions \( u(x,0) = 0, \ u_t(x,0) = h(|x|) \), where \( h(r) = H(1-r) \) for \( r > 0 \), where \( H \) is the heaviside function. Plot the solution \( u(0,t) \).

3 The Diffusion Equation (due April 11, 2016)

1. Suppose that particles in discrete boxes of size \( \Delta x \) leave box \( j \) to box \( j \pm 1 \) at the rate \( \frac{\lambda_{j+1}}{2\Delta x^2} \), where \( \lambda_j = \lambda(j\Delta x) \) for some smooth function \( \lambda(x) \). Derive the limiting diffusion equation, written in conservation form. Identify the different flux terms.

2. A simple model for the dispersal of seeds is that once they become airborne, they diffuse and advect with the wind and drop onto the ground at a linear rate. Thus, the density of seeds in the air (in a one dimensional region) is specified by \( u \) where

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - v \frac{\partial u}{\partial x} - ku, \]

and the amount of seed on the ground is determined by \( q \) where

\[ \frac{\partial q}{\partial t} = ku, \]

starting from initial data \( u(x,0) = \delta(x), \ q(x,0) = 0 \).

Find \( q(x,\infty) = \lim_{t \to \infty} q(x,t) \), and plot it for \( v = 0 \) and \( v > 0 \), using nondimensional variables.

3. (a) Find solutions of the diffusion equation \( u_t = u_{xx} \) of the form \( u(x,t) = U \left( \frac{x}{\sqrt{t}} \right) \) expressed in terms of the error function

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x^2)dx. \]
(b) Use this solution to solve the equation $u_t = Du_{xx}$ on the domain $x > 0, t > 0,$ subject to conditions $u(0, t) = 1$ and $u(x, 0) = 0$ for $x > 0$.

(c) Find and plot the curve $x = X(t)$ along which $u(X(t), t) = \frac{1}{2}$.

(d) Calculate the total amount of $u$ for $x > 0$, $\int_0^\infty u(x, t) dx$, as a function of $t$.

4. Suppose the population of some organism is governed by the equation

$$u_t = Du_{xx} + ku(1 - u)$$

(12)

on the interval $0 < x < L$ subject to boundary conditions $u_x = 0$ at $x = 0$, and $Du_x + \alpha u = 0$ at $x = L$. Under what conditions on the parameters $D$, $L$, $k$; and $\alpha$ can such a population survive?

4. **Laplace’s Equation (due April 28, 2016)**

1. Find the solution $u(r, \theta)$ of Laplace’s equation $\nabla^2 u = 0$ on the interior of a circular domain of radius $R$ subject to Dirichlet boundary data $u(R, \theta) = \cos(n \theta)$ for any integer $n$.

2. Find the solution $u(r, \theta)$ of Laplace’s equation $\nabla^2 u = 0$ on the exterior of a circular domain of radius $R$ subject to Dirichlet boundary data $u(R, \theta) = \cos(n \theta)$ for any integer $n$.

3. Under what conditions on $\lambda$ does a solution $u(r, \theta)$ of

$$\nabla^2 u = -1,$$

(13)

on the annulus $1 < r < 2$, with boundary conditions $u_\nu = \cos \theta$ at $r = 1$, and $u_\nu = \lambda \cos^2 \theta$ at $r = 2$, where $u_\nu$ refers to the outward normal derivative? Find a solution when it exists. Is it unique?

Hint: Make use of Green’s integral identity $\int_{\Omega} (v \nabla^2 u - u \nabla^2 v) dx = \int_{\partial \Omega} (vu_\nu - uv_\nu) d\sigma$ with $v = 1$.

4. Find the Green’s function for a two dimensional disc of radius $R$.

5. Find the Green’s function for the unit hemisphere in three dimensions $x^2 + y^2 + z^2 \leq 1$, $z > 0$. 

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