Math 6420
Homework Exercises

1 First Order PDE's (due Feb. 17, 2016)

1. Find the solution of the problem

$$u_t + cu_x = f(x,t), \qquad u(x,0) = 0, \tag{1}$$

where $f(x,t) = \exp(-t)\sin(x)$.

Solution

In characteristic variables, the equation is

$$\frac{du}{dt} = \exp(-t)\sin(x), \qquad \frac{dx}{dt} = c, \tag{2}$$

so that $x = x_0 + ct$ and

$$\frac{du}{dt} = \exp(-t)\sin(x_0 + ct). \tag{3}$$

Integrating this and applying the initial condition yields

$$u(t,x_0) = \frac{1}{1+c^2}(c\cos(x_0)+\sin(x_0)-c\exp(-t)\cos(ct+x_0)-\exp(-t)\sin(ct+x_0)), \tag{4}$$

which, when x_0 is eliminated yields

$$u(t,x) = \frac{1}{1+c^2}(c\cos(x-ct)+\sin(x-ct)-c\exp(-t)\cos(x)-\exp(-t)\sin(x)). \tag{5}$$

2. Solve the Burgers equation $u_t + uu_x = 0$ with initial data

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}. \tag{6}$$

Solution

According to the method of characteristics, $u = u_0$ along the straight lines $\frac{dx}{dt} = u_0$. Thus,

- $u = 1$ along curves $x = x_0 + t$ for $x_0 < 0$,
- $u = 1 - x_0$ along curves $x = (1-x_0)t + x_0$ for $0 < x_0 < 1$,
- $u = 0$ along curves $x = x_0$ for $x_0 > 1$.

This is great except for the fact that this solution is multivalued. In particular, at $t = 1$ there is a convergence of characteristics at the point $x = 1$. Beyond here there must be a shock, travelling with speed $\dot{S}$

$$\dot{S} = \frac{q(u_-) - q(u_+)}{u_- - u_+} = \frac{1}{2}(u_- - u_+) = \frac{1}{2}, \tag{7}$$

so that $X(t) = 1 + \frac{1}{2}(t-1)$, for $t > 1$. Now the solution is uniquely determined with $u = 1$ for $x < X(t)$ and $u = 0$ for $x > X(t)$.

3. A reasonable model for automobile speed in a long single lane tunnel is

$$v(u) = v_m \cdot \begin{cases} 1 & 0 \le u \le u_c, \\ \frac{\ln(\frac{u_m}{u})}{\ln(\frac{u_m}{u_c})} & u_c \le u \le u_m, \end{cases} \tag{8}$$

where u is the density of cars. Typical parameter values are $u_c = 7$car/km, $v_m = 90$km/h, $u_m = 110$car/km, $\ln(\frac{u_m}{u_c}) = 2.75$.

Suppose the initial density is

$$u = \begin{cases} u_m & x < 0 \\ 0 & x > 0 \end{cases}. \tag{9}$$

Determine the trajectory in the x, t plane of a car starting at position $x = x_0 < 0$ and determine the time it takes for the car to enter the tunnel, and for it to pass through a tunnel of length 5 km.

Solution We need to sort out the structure of the expansion fan in the vicinity of $x = 0$. We start by observing that $q(u) = uv(u)$ and $q'(u) = v(u) + uv'(u)$,

$$q'(u) = v_m \cdot \begin{cases} 1 & 0 \le u \le u_c, \\ \frac{\ln(\frac{u_m}{u})-1}{\ln(\frac{u_m}{u_c})} & u_c \le u \le u_m, \end{cases} \tag{10}$$

This leads to four solution regimes.

- $u = u_m$ for $x < tq'(u_m)$,
- $u = u_0$ on $x = tq'(u_0)$ for $u_c \le u_0 \le u_m$,
- $u = u_c$ for $q'(u_c^+) < \frac{x}{t} < q'(u_c^-)$,
- $u = 0$ for $x > tq'(u_c^-)$.

We invert the relationship for the second region, to find

$$x\frac{\ln(\frac{u_m}{u_c})}{tv_m} + 1 = \ln(\frac{u_m}{u_0}), \tag{11}$$

so that

$$v = \frac{x}{t} + \frac{v_m}{\ln(\frac{u_m}{u_c})}, \tag{12}$$

provided

$$tq'(u_m) < x < q'(u_c^+), \tag{13}$$

which is

$$-t\frac{v_m}{\ln(\frac{u_m}{u_c})} < x < tv_m(1 - \frac{1}{\ln(\frac{u_m}{u_c})}). \tag{14}$$

Now suppose a car starts at position $x_0 < 0$ at time $t = 0$. Until time $t_0 = -x_0\frac{\ln(\frac{u_m}{u_c})}{v_m}$, the velocity is $v = 0$. After this, the car path is

$$\frac{dx}{dt} = \frac{x}{t} - \frac{x_0}{t_0}, \tag{15}$$

which is the path

$$x = \frac{x_0}{t_0}t(1 - \ln(\frac{t}{t_0})). \tag{16}$$

Clearly, this hits $x = 0$ at $t = t_0 e$.

Now, let's find out how long it takes to get through the tunnel. The time that the car enters region 3 is when

$$x = \frac{x_0}{t_0}t(1 - \ln(\frac{t}{t_0})) = tv_m(1 - \frac{1}{\ln(\frac{u_m}{u_c})}), \tag{17}$$

or

$$t = t_0\frac{u_m}{u_c}. \tag{18}$$

When this happens the car is at position $x = x_0\frac{u_m}{u_c}(1 - \ln(\frac{u_m}{u_c}))$. After that the car travels at velocity v_m. To travel through a tunnel of length d, the car must travel total distance $d - x_0$. Thus the total travel time is

$$\begin{aligned} T &= \frac{1}{v_m}\left(d - x_0 - x_0\frac{u_m}{u_c}(1 - \ln(\frac{u_m}{u_c}))\right) + t_0\frac{u_m}{u_c} && (19)\\ &= \frac{1}{v_m}\left(d - x_0 - x_0\frac{u_m}{u_c}\right) && (20) \end{aligned}$$

which, using numbers given above yields

$$T = \frac{1}{90}(5 - 16.6x_0)\text{h}. \tag{21}$$

4. Determine the solvability conditions for the linear problem

$$a(x,y)u_x - u_y = -u, \qquad u(x,x^2) = g(x). \tag{22}$$

Examine the specific case $a(x,y) = \frac{y}{2}$, $g(x) = \exp(-\gamma x^2)$.

Solution

In characteristic form the equation is

$$\frac{dx}{dt} = a(x,y), \qquad \frac{dy}{dt} = -1, \qquad \frac{du}{dt} = -u, \tag{23}$$

with initial data ($t=0$) specified as $u(x_0, x_0^2) = g(x_0)$. These initial data uniquely determine the solution provided $2x_0 a(x_0, x_0^2) \neq -1$ for all x_0.

At a point where this condition fails we require that the matrix

$$J = \begin{pmatrix} a(x_0, x_0^2) & -1 & -g(x_0) \\ 1 & 2x_0 & g'(x_0) \end{pmatrix}, \tag{24}$$

have rank 1. If this matrix has rank 2, solutions do not exist.

For the specific case at hand, $a(x,y) = \frac{y}{2}$ and the Cauchy condition is $x_0^3 \neq -1$ which of course, fails if $x_0 = -1$. Further,

$$J = \begin{pmatrix} \frac{x_0^2}{2} & -1 & -\exp(-\gamma x_0^2) \\ 1 & 2x_0 & 2\gamma \exp(-\gamma x_0^2) \end{pmatrix}, \tag{25}$$

which at $x_0 = -1$ is

$$J = \begin{pmatrix} \frac{1}{2} & -1 & -\exp(-\gamma) \\ 1 & -2 & 2\gamma \exp(-\gamma) \end{pmatrix}, \tag{26}$$

and it has rank 1 only if $\gamma = -1$. Hence the problem is well-posed if $\gamma = 2$ and not if $\gamma \neq -1$.

5. Solve the Cauchy problem

$$u_x^2 + u_y^2 = 4u, \qquad u(x,0) = x^2 \tag{27}$$

Solution

Take $F = p^2 + q^2 - 4u$. In characteristic form

$$\frac{dx}{dt} = 2p, \qquad \frac{dy}{dt} = 2q, \qquad \frac{dp}{dt} = 4p, \qquad \frac{dq}{dt} = 4q, \qquad \frac{du}{dt} = 8u. \tag{28}$$

The solution is

$$\begin{aligned} u(t) &= x_0^2 \exp(8t), && (29) \\ p(t) &= p_0 \exp(4t), \qquad q(t) = q_0 \exp(4t), && (30) \\ x(t) &= \frac{p_0}{2}(\exp(4t) - 1) + x_0, \qquad y(t) = \frac{q_0}{2}(\exp(4t) - 1). && (31) \end{aligned}$$

Consistency for initial data requires

$$p_0^2 + q_0^2 - 4x_0^2 = 0, \qquad p_0 = 2x_0 \tag{32}$$

so that

$$\begin{aligned} u(t) &= x_0^2 \exp(8t), && (33) \\ p(t) &= 2x_0 \exp(4t), \qquad q(t) = 0, && (34) \\ x(t) &= x_0(\exp(4t) - 1) + x_0, \qquad y(t) = 0. && (35) \end{aligned}$$

which reduces to $u(x,y) = x^2$.

2 The Wave Equation (due March 21, 2016)

1. Solve the problem

$$u_{tt} = u_{xx}, \qquad 0 < x < 1, \qquad t > 0, \tag{36}$$

subject to initial conditions $u(x,0) = u_t(x,0) = 0$ and boundary conditions $u_x(0,t) = 1$, $u(1,t) = 0$.

Solution

Set $u = x - 1 + v$ and then solve for v using separation of variables to find

$$v(x,t) = \sum_n (a_n \cos(\lambda_n t)\cos(\lambda_n x) + b_n \sin(\lambda_n t)\cos(\lambda_n x)) \tag{37}$$

where $\lambda_n = (2n+1)\frac{\pi}{2}$, subject to initial data $v_t(x,0) = 0$, $v(x,0) = 1-x$. Consequently, $b_n = 0$ and

$$\sum_n a_n \cos(\lambda_n x) = 1 - x \tag{38}$$

so that

$$a_n = \frac{8}{\pi^2(2n+1)^2}, \tag{39}$$

$$v(x,t) = \sum_n \frac{8}{\pi^2(2n+1)^2}\cos((2n+1)\frac{\pi}{2}t)\cos((2n+1)\frac{\pi}{2}x). \tag{40}$$

2. Find the characteristics for the equation $u_{tt} = tu_{xx}$.

Solution

Characteristics are level surfaces of the of ϕ and ψ where

$$\phi_t - \sqrt{t}\phi_x = 0, \qquad \psi_t + \sqrt{t}\psi_x = 0, \tag{41}$$

which are the curves $\frac{dx}{dt} = \pm\sqrt{t}$, or

$$x = x_0 \pm \frac{2}{3}t^{\frac{3}{2}}. \tag{42}$$

3. Is the problem

$$u_{tt} = u_{xx}, \qquad -t < x < t, \tag{43}$$

subject to conditions $u(x,x) = f(x)$, $u(x,-x) = g(x)$ with $f(0) = g(0)$ well posed? If it is well posed, find the solution.

Solution

The problem is well posed. The unique solution is

$$u(x,t) = g(\frac{x-t}{2}) - g(0) + f(\frac{x+t}{2}). \tag{44}$$

4. Find characteristics and use these to find the general solution of $t^2u_{tt}+2tu_{xt}+u_{xx}-u=0$.

   Solution

   This equation is parabolic, so there is only one family of characteristics, satisfying

$$t\phi_t + \phi_x = 0, \tag{45}$$

   with solutions $\phi = \ln t - x = $ constant. Introduce the change of variables

$$\xi = \phi = \ln t - x, \qquad \eta = x, \tag{46}$$

   and calculate that

$$u_{tt} = \frac{u_{\xi\xi}}{t^2} - \frac{u_\xi}{t^2}, \qquad u_{xt} = -\frac{u_{\xi\xi}}{t} + \frac{u_{\xi\eta}}{t}, \qquad u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \tag{47}$$

   so that the equation becomes

$$u_\xi = u_{\eta\eta} - u. \tag{48}$$

   The fundamental solution of this problem is

$$u(\xi,\eta) = \frac{u_0}{\sqrt{\xi}}\exp(-\frac{\eta^2}{\xi} - \xi). \tag{49}$$

5. Solve the problem $u_{tt} - c^2\nabla^2 u = 0$ in two and three dimensional space for $t > 0$ subject to initial conditions $u(\mathbf{x},0) = 0, u_t(\mathbf{x},0) = h(|\mathbf{x}|)$, where $h(r) = H(1-r)$ for $r > 0$, where H is the heaviside function. Plot the solution $u(0,t)$.

3 The Diffusion Equation (due April 11, 2016)

1. Suppose that particles in discrete boxes of size Δx leave box j to box $j \pm 1$ at the rate $\frac{\lambda_{j\pm1}}{\Delta x^2}$, where $\lambda_j = \lambda(j\Delta x)$ for some smooth function $\lambda(x)$. Derive the limiting diffusion equation, written in conservation form. Identify the different flux terms.

   Solution

   The balance equation for the i^{th} box is

$$\begin{aligned}\frac{du_i}{dt} &= \frac{1}{\Delta x^2}\Big(\lambda_i u_{i-1} - \lambda_{i-1}u_i - \lambda_{i+1}u_i + \lambda_i u_{i+1}\Big),\\ &= \frac{1}{\Delta x^2}\Big(\lambda_i(u_{i-1} - 2u_i + u_{i+1}) - (\lambda_{i-1} - 2\lambda_i + \lambda_{i+1}u_i)\Big),\\ &\approx \lambda(x)\frac{\partial^2 u}{\partial x^2} - \lambda''(x)u + O(\Delta x^2),\\ &= \frac{\partial}{\partial x}\Big((\lambda(x)\frac{\partial u}{\partial x} - \lambda'(x)u\Big) + O(\Delta x^2),\end{aligned} \tag{50}$$

   which leads to the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\lambda(x)\frac{\partial u}{\partial x} - \lambda'(x)u\Big). \tag{51}$$

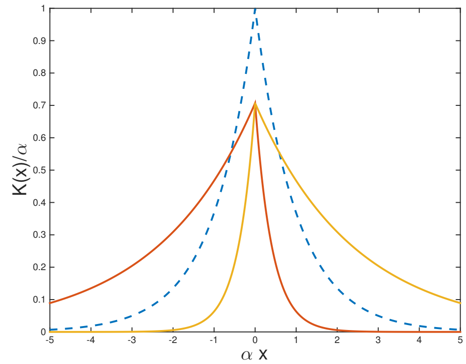

Figure 1: Plot of the dispersal kernel $\frac{q(x,\infty)}{\alpha}$ plotted as a function of αx, $\alpha = \sqrt{\frac{k}{D}}$ in the case with a steady wind $\beta = \frac{v}{\sqrt{Dk}} = -2$ (solid curve, left), $\beta = 0$ (dashed curve), and $\beta = 2$ (solid curve, right).

2. A simple model for the dispersal of seeds is that once they become airborne, they diffuse and advect with the wind and drop onto the ground at a linear rate. Thus, the density of seeds in the air (in a one dimensional region) is specified by u where

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} - ku, \tag{52}$$

and the amount of seed on the ground is determined by q where

$$\frac{\partial q}{\partial t} = ku, \tag{53}$$

starting from initial data $u(x,0) = \delta(x)$, $q(x,0) = 0$.

Find $q(x,\infty) = \lim_{t\to\infty} q(x,t)$, and plot it for $v = 0$ and $v > 0$, using nondimensional variables.

Solution Rescale space and time so that seed dispersal is governed by the non-dimensional equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial x} - u, \tag{54}$$

with $\beta = \frac{v}{\sqrt{Dk}}$.

The solution of this is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\exp(-\frac{(x-\beta t)^2}{4t} - t), \tag{55}$$

and

$$q(x,\infty) = \int_0^\infty u(x,t)dt = \frac{1}{\sqrt{\beta^2+4}}\exp\Big(-\frac{1}{2}(\sqrt{\beta^2+4}|x| - \beta x)\Big). \tag{56}$$

A plot of the resulting dispersal kernel is shown in Fig. 1.

3. (a) Find solutions of the diffusion equation $u_t = u_{xx}$ of the form $u(x,t) = U(\frac{x}{\sqrt{t}})$ expressed in terms of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x^2)dx. \tag{57}$$

(b) Use this solution to solve the equation $u_t = Du_{xx}$ on the domain $x > 0$, $t > 0$, subject to conditions $u(0,t) = 1$ and $u(x,0) = 0$ for $x > 0$.

(c) Find and plot the curve $x = X(t)$ along which $u(X(t),t) = \frac{1}{2}$.

(d) Calculate the total amount of u for $x > 0$, $\int_0^\infty u(x,t)dx$, as a function of t.

Solution

(a) With $u(x,t) = U(\xi)$ where $\xi = \frac{x}{\sqrt{t}}$, it must be that $U'' + \frac{1}{2}\xi U' = 0$ so that

$$U' = W_0 \exp(-\frac{\xi^2}{4}) \tag{58}$$

and

$$U(\xi) = U_0 + W_0 \int_0^\xi \exp(-\frac{\eta^2}{4})d\eta = U_0 + V_0 \mathrm{erf}(\frac{\xi}{2}). \tag{59}$$

(b) To match boundary and initial data take

$$u(x,t) = 1 - \mathrm{erf}(\frac{\xi}{2}), \qquad \xi = \frac{x}{\sqrt{Dt}}. \tag{60}$$

(c) $u(x(t),t) = \frac{1}{2}$ along the curve $\frac{x}{\sqrt{Dt}} = 0.956$ which is a parabola.

(d) Calculate using integration by parts that

$$\int_0^\infty u(x,t)dx = 2\sqrt{\frac{Dt}{\pi}}. \tag{61}$$

4. Suppose the population of some organism is governed by the equation

$$u_t = Du_{xx} + ku(1-u) \tag{62}$$

on the interval $0 < x < L$ subject to boundary conditions $u_x = 0$ at $x = 0$, and $Du_x + \alpha u = 0$ at $x = L$. Under what conditions on the parameters D, L, k, and α can such a population survive?

Solution

It suffices to do a linearized analysis, examining the stability of the solution $u = 0$, by solving

$$u_t = Du_{xx} + ku. \tag{63}$$

The solution is

$$u(x,t) = \exp(\lambda t)\cos(ax), \tag{64}$$

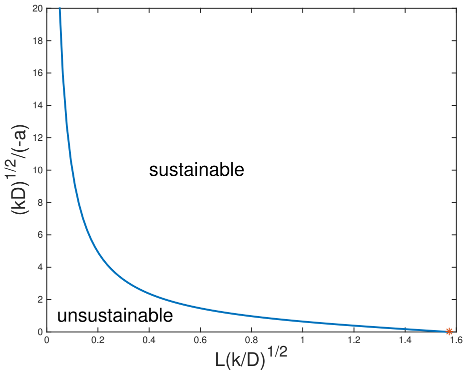

Figure 2: Plot of the critical curve (65).

where $aD\tan(aL) = \alpha$, and $\lambda = k - Da^2$, and sustainability occurs when $\lambda > 0$. To make physical sense, it must be that $\alpha > 0$.

One way to represent the curve $\lambda = 0$ is parametrically via

$$\frac{\alpha}{\sqrt{kD}} = \tan\beta, \qquad L\sqrt{\frac{k}{D}} = \beta \tag{65}$$

This curve is shown plotted in Fig. 2.

4 Laplace's Equation (due April 28, 2016)

1. Find the solution $u(r,\theta)$ of Laplace's equation $\nabla^2 u = 0$ on the interior of a circular domain of radius R subject to Dirichlet boundary data $u(R,\theta) = \cos(n\theta)$ for any integer n.

   Solution

   Solve Laplace's equation in polar coordinates with separation of variables by assuming a solution of the form $u(r,\theta) = R(r)T(\theta)$ and require

   $$r(rU')' - \lambda^2 R = 0, \qquad T'' + \lambda^2 T = 0 \tag{66}$$

   so that $T(\theta) = \cos(n\theta), \qquad R(r) = Ar^n$ and consequently,

   $$u(r,\theta) = (\frac{r}{R}\Big)^n \cos(n\theta) \tag{67}$$

2. Find the solution $u(r,\theta)$ of Laplace's equation $\nabla^2 u = 0$ on the exterior of a circular domain of radius R subject to Dirichlet boundary data $u(R,\theta) = \cos(n\theta)$ for any integer n.

   Solution

   Using the same methodology as above, find that

   $$u(r,\theta) = (\frac{R}{r}\Big)^n \cos(n\theta) \tag{68}$$

3. Under what conditions on λ does a solution $u(r,\theta)$ of

$$\nabla^2 u = -1, \tag{69}$$

on the annulus $1 < r < 2$, with boundary conditions $u_\nu = \cos\theta$ at $r = 1$, and $u_\nu = \lambda\cos^2\theta$ at $r = 2$, where u_ν refers to the outward normal derivative? Find a solution when it exists. Is it unique?

Hint: Make use of Green's integral identity $\int_\Omega (v\nabla^2 u - u\nabla^2 v)dx = \int_{\partial\Omega}(vu_\nu - uv_\nu)d\sigma$ with $v = 1$.

Solution

Start by setting $u(r,\theta) = v(r,\theta) - \frac{r^2}{4}$ and then require $\nabla^2 v = 0$ with boundary conditions on v, $\frac{\partial v}{\partial r}|_{r=2} = 1 + \lambda\cos^2(\theta)$, $\frac{\partial v}{\partial r}|_{r=1} = \frac{1}{2} - \cos\theta$.

Using separation of variables we set

$$v(r,\theta) = A_0 + B_0 \ln r + (A_1 r + \frac{B_1}{r})\cos\theta + (A_2 r^2 + \frac{B_2}{r^2})\cos(2\theta), \tag{70}$$

for which

$$\frac{\partial v}{\partial r} = \frac{B_0}{r} + (A_1 - \frac{B_1}{r^2})\cos\theta + (2A_2 r - 2\frac{B_2}{r^3})\cos(2\theta), \tag{71}$$

which means we must require

$$\frac{B_0}{2} + (A_1 - \frac{B_1}{4})\cos\theta + (4A_2 - 2\frac{B_2}{8})\cos(2\theta) = 1 + \frac{\lambda}{2} + \frac{\lambda}{2}\cos(2\theta), \tag{72}$$

and

$$B_0 + (A_1 - B_1)\cos\theta + (2A_2 - 2B_2)\cos(2\theta) = \frac{1}{2} - \cos\theta \tag{73}$$

We see quickly that the solution has $B_0 = \frac{1}{2}$ and this is a solution if and only if $1 + \frac{\lambda}{2} = \frac{1}{4}$, i.e., if and only if $\lambda = \frac{3}{2}$. Then we determine the remaining coefficients, finally finding $A_1 = \frac{1}{3}$, $B_1 = \frac{4}{3}$, $A_2 = B_2 = -\frac{1}{5}$,

$$u(r,\theta) = A_0 + \frac{1}{2}\ln r + \frac{1}{3}(r + \frac{4}{r})\cos\theta - \frac{1}{5}(r^2 + \frac{1}{r^2})\cos(2\theta) - \frac{r^2}{4}, \tag{74}$$

where A_0 is arbitrary.

4. Find the Green's function for a two dimensional disc of radius R.

Solution

Use that the fundamental solution for Laplace's equation in two dimensions is $G = \frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{y}|$ to write that

$$G(r,\theta,r_0,\phi) = \frac{1}{2\pi}\ln\Big(R^2 \frac{r^2 + r_0^2 - 2rr_0\cos(\theta-\phi)}{r^2 r_0^2 + R^4 - 2rr_0\cos(\theta-\phi)}\Big). \tag{75}$$

5. Find the Green's function for the unit hemisphere in three dimensions $x^2+y^2+z^2 \leq 1$, $z > 0$.

Solution

Use the method of images to write

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{R}{|\mathbf{x}|}\frac{1}{|\mathbf{x}^*-\mathbf{y}|} - \frac{1}{|\mathbf{x}_- -\mathbf{y}|} + \frac{R}{|\mathbf{x}|}\frac{1}{|\mathbf{x}_-^*-\mathbf{y}|}\right), \qquad (76)$$

where $\mathbf{x}_-$ reverses the sign of the z-component of $\mathbf{x}$, and $\mathbf{x}^* = \frac{R^2}{|\mathbf{x}|^2}\mathbf{x}$.