Homework Exercise 1 for Mathematics 6410 - Fall 2020 (due Set. 16, 2020)

1. A simple population growth model with harvesting (with an "Allee effect") is

$$\frac{du}{dt} = \alpha u(A-u)(u-a) - hu, \tag{1}$$

where $0 < a < A$, $h \geq 0$.

   (a) Nondimensionalize this equation in such a way (with nondimensional variables U and τ) that the region $0 < U < 1$ is invariant when $h = 0$. What (and how many) are the non-dimensional parameters for this problem?

   (b) Give a full graphical analysis of this equation. For what values of harvesting rate is the population sustainable?

   (c) Find an implicit relationship for the solution U as a function of τ when $U(0) = 1$.

2. Solve the differential equation $\frac{du}{dt} = -t\sqrt{u}$ for $u(0) = u_0 > 0$, and $t > 0$. Show that all solutions (i.e., for all $u_0 > 0$) eventually intersect.

3. Find the general solution of the differential equation

$$\frac{dx}{dt} = \frac{3x - 2t}{t} \tag{2}$$

4. (a) Find an integral curve for the equation $x\frac{dy}{dx} + 3x - 2y = 0$.

   (b) Show that all first order linear differential equations (of the form $f(x)\frac{dy}{dx} + g(x)y = h(x)$) have integral curves of the form $A(x)y + B(x) = y_0$. What are $A(x)$ and $B(x)$?

5. Suppose $h(t)$ is a periodic function, $h(t+1) = h(t)$. Under what conditions on a and $h(t)$ does the equation

$$\frac{dx}{dt} = ax + h(t) \tag{3}$$

have a *stable* periodic solution?

6. Show that the differential equation $\frac{du}{dt} = u(A(t) - u)$ with $0 < A(t) \leq 1$, $A(t+1) = A(t)$ and $A(t) \neq 1$, has a unique, positive, stable periodic solution.

   Hint: Define the Poincare map $P(u_0) = u(1, u_0)$, where $u(t, u_0)$ satisfies the differential equation with $u(0, u_0) = u_0$. Show that $P(1) < 1$, $P'(0) > 1$, and $P''(u_0) < 0$ for $0 \leq u_0 \leq 1$.