Homework 3: Due March 24, 2015

1. The purpose of this exercise is to demonstrate that the bistable equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)
\]  

with \( f(u) = u(u - \alpha)(1 - u) \) and \(-\infty < x < \infty\) exhibits threshold behavior. For this exercise take \( \alpha = 0.1 \).

(a) Find a nontrivial steady solution of this problem, i.e., a solution of \( \frac{\partial^2 u}{\partial x^2} + f(u) = 0 \) for which \( \lim_{x \to \pm\infty} u(x) = 0 \). Plot it both in the phase plane and as a function of \( x \).

(b) Solve (i.e., simulate numerically) the bistable equation starting from initial data which are scalar multiples of the steady solution found in the previous part. Take the scalar multiple to be slightly larger than one and slightly smaller than one. What is the eventual fate of the different solutions?

2. Morris and Lecar (1981) proposed the following two-variable model of membrane potential for a barnacle muscle fiber:

\[
C_m \frac{\partial V}{\partial t} + I_{\text{ion}}(V, W) = I_{\text{app}} + D \frac{\partial^2 V}{\partial x^2},
\]

\[
\frac{\partial W}{\partial t} = \phi \Lambda(V)[W_{\infty}(V) - W],
\]

where \( V \) = membrane potential, \( W \) = fraction of open K\(^+\) channels, \( C_m \) = membrane capacitance, \( I_{\text{app}} \) = externally applied current, \( \phi \) = maximum rate for closing K\(^+\) channels, and

\[
I_{\text{ion}}(V, W) = g_{Ca}M_{\infty}(V)(V - V_{Ca}^0) + g_KW(V - V_K^0) + g_L(V - V_L^0),
\]

\[
M_{\infty}(V) = \frac{1}{2} \left( 1 + \tanh \left( \frac{V - V_1}{V_2} \right) \right),
\]

\[
W_{\infty}(V) = \frac{1}{2} \left( 1 + \tanh \left( \frac{V - V_3}{V_4} \right) \right),
\]

\[
\Lambda(V) = \cosh \left( \frac{V - V_3}{2V_4} \right).
\]

Typical rate constants in these equations are shown in Table 5.1.

(a) Make a phase portrait for the Morris–Lecar differential equations (with \( D = 0 \)). Plot the nullclines and show some typical trajectories, demonstrating that the model is excitable.

(b) Find travelling wave solutions of the Morris-Lecar equations numerically. Plot these solutions in the \( V - W \) phase plane. Estimate the wave speed as a function of \( D \). Verify (numerically) that the speed scales like \( \sqrt{D} \).
\begin{table}
\begin{tabular}{|l|l|}
\hline
\(C_m = 20 \, \mu F/cm^2\) & \(I_{\text{app}} = 0.06 \, mA/cm^2\) \\
\(g_{Ca} = 4.4 \, mS/cm^2\) & \(g_K = 8 \, mS/cm^2\) \\
\(g_L = 2 \, mS/cm^2\) & \(\phi = 0.04 \, (ms)^{-1}\) \\
\(V_1 = -1.2 \, mV\) & \(V_2 = 18 \, mV\) \\
\(V_3 = 2\) & \(V_4 = 30 \, mV\) \\
\(V^0_{Ca} = 120 \, mV\) & \(V^0_K = -84 \, mV\) \\
\(V_L = -60 \, mV\) & \\
\hline
\end{tabular}
\end{table}

Table 1: Typical parameter values for the Morris–Lecar model.