Math 5110 - Fall 2007
Final Exam
due 5:00 pm. Dec. 11, 2007

For this exam you may consult your notes, but not your friends, neighbors or fellow students. The following problems are worth 25 points each for a total of 100 points.

1. A model for a population of insects with discrete generations is

$$N_{n+1} = \exp(A)N_n \exp(-bN_n) \tag{1}$$

with $b > 0$.

(a) Show that this model can be reduced to the model with a single parameter

$$u_{n+1} = u_n \exp(a - u_n) \equiv f(u_n). \tag{2}$$

What change of variables accomplishes this?

(b) For what values of a is the origin a stable attractor, and for what values of a is the origin unstable?

(c) For what values of a is there a positive steady solution?

(d) For what values of a is the positive steady solution stable, and for what values of a is the positive steady solution unstable?

(e) For what values of a is the approach to the stable positive solution monotone?

(f) For what values of a is the approach to the stable positive solution oscillatory?

(g) For what values of a is the (initial) departure from the unstable positive solution oscillatory?

(h) For what values of a is the (initial) departure from the unstable positive steady solution monotone?

(i) (5 pts Extra Credit - This is a bit harder than other parts) Suppose $a > 2$. Prove that there must be a period two solution. Hint: Sketch the function $f(f(u))$ and determine the slope of this function at the positive fixed point of $f(u)$.

2. The population for the fish in a pond is believed to satisfy the differential equation

$$\frac{dx}{dt} = Ax(1 - \frac{x}{K}) - hx, \tag{3}$$

where h represents the total harvesting effort.

(a) For what values of h is there a stable steady positive population? What is the steady rate of harvest for these values of effort?

(b) At what amount of effort is the rate of harvest maximized?

(c) Suppose that Walt fishes at the lake every day expending effort h_1, and you plan to expend effort h_2. What will be your (steady state) rate of harvesting?

(d) A what rate of effort h_2 will you maximize your personal harvest rate? If you optimize your personal harvest rate, is the total effort sub-optimal, super-optimal, or exactly optimal?

3. A proposed model for the production of a substance called autoinducer is

$$\frac{dA}{dt} = \epsilon + \frac{\alpha A^2}{1+A^2} - \beta(A-E), \tag{4}$$

$$(1-\rho)\frac{dE}{dt} = \rho\beta(A-E) - (1-\rho)\delta E, \tag{5}$$

where ϵ is very small and $\frac{\alpha}{\beta} < 2$. Here A is the concentration of autoinducer in cells, E is the concentration of autoinducer in the medium surrounding the cells, and ρ is the density of cells.

(a) How many qualitatively different phase portraits are there? (Sketch them.) For each case, state how many steady state solutions there are and give their stability. (This can be determined graphically with no algebraic calculations!)

(b) Suppose the density of cells changes gradually starting very small but then increases to be large and then gradually decreasing again. Describe how the amount of autoinducer changes during this time.

4. A chemical species A can combine with itself to form a dimer B via the reaction

$$A + A \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} B. \tag{6}$$

The dimer can bind to an enzyme E to activate the enzyme

$$B + E \underset{k_{-2}}{\overset{k_2}{\rightleftarrows}} E^*, \tag{7}$$

and the activated enzyme produces the chemical A

$$\xrightarrow{k_3 E^*} A. \tag{8}$$

Assume that B is not consumed by the binding with E.

(a) Write the differential equations describing the amount of A, B, and E^*. Assume that $E + E^* = E_0$.

(b) Suppose that E^* is always in quasiequilibrium. What is the concentration of E^*?

(c) Suppose that the dimerization reaction is a fast reaction that is always in quasiequilibrium. Find B as a function of A.

(d) Let $y = A + 2B$. Find A as a function of y, and B as a function of y.

(e) Find a single differential equation for the growth of y, $\frac{dy}{dt} = f(y)$. Sketch $f(y)$.