Math 5110 - Fall 2007
Mid Term Exam - 2
Due 5pm Thursday Nov. 15, 2007

For this exam you may use Maple, Matlab and Pplane, however, unless you thoroughly describe your work, no partial credit will be given for incorrect answers. Do not turn in Maple or Matlab code.

The following is a model for three populations in a chemostat.

$$\frac{dS}{d\tau} = d(S_i - S) - \frac{\mu_1}{y_1}SH, \tag{1}$$
$$\frac{dH}{d\tau} = \mu_1 SH - dH - \frac{\mu_2}{y_2}HP, \tag{2}$$
$$\frac{dP}{d\tau} = \mu_2 HP - dP. \tag{3}$$

1. Describe the meaning of each of the terms on the right hand side of these equations. In particular, what is the interaction between these three populations? What assumption is made so that the removal rate d is the same in each equation? How would you describe the relationships between these species?

2. Using a change of variables, this system can be reduced to

$$\frac{dx}{dt} = 1 - x - axy, \tag{4}$$
$$\frac{dy}{dt} = axy - y - byz, \tag{5}$$
$$\frac{dz}{dt} = byz - z. \tag{6}$$

   What is the change of variables that accomplishes this? What are the parameters a and b in terms of original parameters of the problem?

3. Find a differential equation for $s = x+y+z$. What is the solution of this differential equation?

4. Suppose $s = 1$. Find the reduced system of equations for y and z.

   For the remaining questions, you can work with the full system of equations (4-6), or the reduced system of equations.

5. Identify all possible steady state solutions of the system What is the biological interpretation of these steady state solutions?

6. Find a steady solution for which two of the three species are zero. Under what conditions is this solution stable and under what conditions is this solution unstable? Sketch the y-z phase portrait in the case that this solution is stable, assuming $s = 1$.

7. Find a steady solution for which exactly one of the three species is zero. For what parameter values does this solution exist and for what parameter values is this solution stable? Sketch the y-z phase portrait in the case that this solution is stable, assuming $s = 1$.

8. Find a steady solution for which all three of the species are nonzero. For what parameter values does this solution exist and for what parameter values is this solution stable? Sketch the y-z phase portrait for this case, assuming $s = 1$.

9. This third steady solution can have two different types of stability. What are they, and for what parameter values do they occur?

10. Summarize the above stability information in a plot of the a-b parameter space, showing the regions for which each of the different steady state solutions is stable. Are there any regions in which more than one steady solution is stable?