Selected Hints and Solutions

Principles of Applied Mathematics; Transformation and Approximation

James P. Keener

1.1.2; (a) Follows from direct verification.

(b) Follows from a). If the norm is known to be induced by an inner product, then a) shows how to uniquely calculate the inner product.

(c) Suppose $||x|| = (\sum_{k=1}^{n} |x_k|^p)^{1/p}$.

($\Leftarrow$) If $p = 2$, then $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$ induces the norm.

($\Rightarrow$) If the norm is induced by an inner product, then from a)

$$\langle x, y\rangle = \frac{1}{4}\left((\sum_{k=1}^{n} |x_k + y_k|^p)^{2/p} - (\sum_{k=1}^{n} |x_k - y_k|^p)^{2/p} \right). \tag{1}$$

Take $x = (1, 0, 0, \ldots, 0)$, and $y = 0, 1, 0, \ldots, 0)$. Then $\langle x, x\rangle = 1, \langle x, y\rangle = 0$, and $\langle x, x + y\rangle = \frac{1}{4}\left((2^p + 1)^{2/p} - 1\right)$. Since for an inner product, $\langle x, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle$, it must be that $(2^p + 1)^{2/p} = 5$. Since $(2^p + 1)^{2/p}$ is a monotone decreasing function of p which aproaches 1 for large p and is unbounded at the origin, the solution of $(2^p + 1)^{2/p} = 5$ at $p = 2$ is unique. We conclude that $p = 2$.

1.1.5; Observe that with $\beta = \langle x, y\rangle/||y||^2$, $x - \beta y$ is orthogonal to y, so that

$$||x - \alpha y||^2 = ||x - \beta y||^2 + ||(\alpha - \beta)y||^2, \tag{2}$$

which is minimized when $\alpha = \beta$. Clearly, if $x = \alpha y$, then

$$|\langle x, y\rangle|^2 = ||x||^2||y||^2. \tag{3}$$

If so, then we calculate directly that $||x - \beta y||^2 = 0$, so that $x = \beta y$.

1.1.6; (a) $\phi_0 = 1, \phi_1 = x, \phi_2 = x^2 - \frac{1}{3}, \phi_3 = x^3 - \frac{3}{5}x,$

(b) $\phi_0 = 1, \phi_1 = x, \phi_2 = x^2 - \frac{1}{2} = \frac{1}{2}\cos(2\cos^{-1} x), \phi_3 = x^3 - \frac{3}{4}x = \frac{1}{4}\cos(3\cos^{-1} x).$

(c) $\phi_0 = 1, \phi_1 = x - 1, \phi_2 = x^2 - 4x + 2, \phi_3 = x^3 - 9x^2 + 18x - 6.$

(d) $\phi_0 = 1, \phi_1 = x, \phi_2 = x^2 - \frac{1}{2}, \phi_3 = x^3 - \frac{3}{2}x.$

1.1.7; $\phi_0 = 1, \phi_1 = x, \phi_2 = x^2 - \frac{1}{3}, \phi_3 = x^3 - \frac{9}{10}x, \phi_4 = x^4 - \frac{33}{28}x^2 + \frac{27}{140}, \phi_5 = x^5 - \frac{1930}{1359}x^3 + \frac{445}{1057}x.$

1.2.1; (a) Relative to the new basis, $A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) Set $C = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Then the representation of A in the new basis is

$$A' = D^{-1}CAC^{-1}D = \begin{pmatrix} 9 & -10 & -2 \\ 4 & -5 & -2 \\ \frac{23}{3} & -\frac{23}{3} & 0 \end{pmatrix}. \tag{4}$$

1.2.2; (c) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ have the same determinant but are not equivalent.

1.2.3; (a) Notice that if $ABx = \lambda x$, then $BA(Bx) = \lambda(Bx)$.

(b) If $AA^*x = \lambda x$, then $\lambda\langle x, x\rangle = \langle AA^*x, x\rangle = \langle A^*x, A^*x\rangle \geq 0$.

1.2.4; If $Ax = \lambda x$ then $\lambda\langle x, x\rangle = \langle Ax, x\rangle = \langle x, A^T x\rangle = -\langle x, Ax\rangle = -\overline{\lambda}\langle x, x\rangle$.

1.2.5; (a) $R(a) = \{(1, 1, 2)^T, (2, 3, 5)^T\}, N(A) = 0, R(A^*) = R^2, N(A^*) = \{(1, 1, -1)^T\}$.

(b) $R(A) = R(A^*) = R^3, N(A) = N(A^*) = 0$.

1.2.6; (a) $T = \begin{pmatrix} 1 & \frac{1}{5} & 0 \\ \frac{1}{3} & \frac{3}{5} & 1 \\ 0 & 1 & 0 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$

(b) $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

(c) $T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(d) $T = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -\sqrt{3} & \frac{2}{\sqrt{3}} \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}.$

1.2.7; $T = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}.$

1.2.8; If x is in M, then $Px = x$, and if x is in the orthogonal complement of M, the $Px = 0$.

1.3.1; Hint: Minimize $\langle Ax, x \rangle$ with a vector of the form $x^T = (1, -1, z, 0)$.

1.3.2; (a) Prove that if the diagonal elements of a symmetric matrix are increased, then the eigenvalues are increased as well.

(b) Find the eigenvalues and eigenvectors of $B = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 8 \end{pmatrix}$, and use them to estimate the eigenvalues of A.

1.3.3; The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 7 \end{pmatrix}$ has a positive, zero, and negative eigenvalue. Apply 1.3.2; a).

1.4.1; (a) b must be orthogonal to $(1, 1, -1)^T$, and the solution, if it exists, is unique.

(b) The matrix A is invertible, so the solution exists and is unique.

1.4.2; b must be in the range of P, namely M.

1.4.3; ($\Rightarrow$) Suppose A is invertible, and try to solve the equation $\sum \alpha_i \phi_i = 0$. Taking the inner product with ϕ_j, we find $0 = A\alpha$, so that $\alpha = 0$, since the null space of A is zero.

($\Leftarrow$) Suppose $\{\phi_i\}$ form a linearly independent set and that $Ax = 0$. Then $\langle x, Ax \rangle = \langle \sum_i x_i \phi_i, \sum_j x_j \phi_j \rangle = 0$, so that $\sum_i x_i \phi_i = 0$, implying that $x = 0$, so that A is invertible (by the Fredholm Alternative).

1.4.4; Since $\langle Ax, x\rangle = \langle x, A^* x\rangle > 0$ for all $x \neq 0$, the null spaces of A and A^* must be empty. Hence, $\langle b, x\rangle = 0$ for all x in $N(A^*)$ so that $Ax = b$ has a solution. Similarly, the solution is unique since the null space of A is empty.

1.5.1; (a) $A' = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A' = \frac{1}{24}\begin{pmatrix} 7 & 4 & 1 \\ 1 & 4 & 7 \end{pmatrix}$

(c) $A' = \frac{1}{12}\begin{pmatrix} 3 & 1 \\ 3 & 5 \\ 0 & -4 \end{pmatrix}$

(d) $A' = \frac{1}{2}\begin{pmatrix} -6 & 2 & 2 \\ -5 & 4 & 1 \\ -4 & 2 & 2 \end{pmatrix}$

1.5.4; $Q = (\phi_1, \phi_2, \phi_3)$, where $\phi_1 = \frac{1}{\sqrt{101}}\begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}$, $\phi_2 = \frac{1}{\sqrt{505}}\begin{pmatrix} 9 \\ 18 \\ -10 \end{pmatrix}$, $\phi_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, and

$$R = \begin{pmatrix} \sqrt{101} & \frac{19}{\sqrt{101}} & \frac{28}{\sqrt{101}} \\ 0 & \frac{35}{\sqrt{505}} & \frac{25}{\sqrt{505}} \\ 0 & 0 & \sqrt{5} \end{pmatrix}.$$

1.5.7; For $A = \begin{pmatrix} 1.002 & 0.998 \\ 1.999 & 2.001 \end{pmatrix}$, singular values are $\sqrt{10}$ and $\epsilon\sqrt{10}$ with $\epsilon = 0.001\sqrt{10}$, $A' = \begin{pmatrix} 0.1 + \frac{2}{\epsilon\sqrt{10}} & 0.2 - \frac{1}{\epsilon\sqrt{10}} \\ 0.1 - \frac{2}{\epsilon\sqrt{10}} & 0.2 + \frac{1}{\epsilon\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 200.1 & -99.8 \\ -199.8 & 100.2 \end{pmatrix}$. Using instead singular values $\sqrt{10}$ and 0, $A' = \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{pmatrix}$.

1.5.9; $A = P\Sigma Q$ where $P = \frac{1}{\sqrt{10}}\begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 2\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, so that $A = \begin{pmatrix} \frac{\sqrt{5}}{20} & \frac{1}{30} & \frac{2}{30} \\ -\frac{\sqrt{5}}{20} & \frac{1}{30} & \frac{2}{30} \end{pmatrix}$.

1.5.12; Assuming real vectors, and using Lagrange multipliers, one finds normal equations, $(YY^T + M)A^T = XY^T, A^TA = I$, where M is an arbitrary symmetric matrix. This system of nonlinear equations has no obvious easy solution.

2.1.3; For $x_n = \sum_{k=1}^{n} \frac{1}{k!}, |x_n - x_m| = \sum_{k=n+1}^{m} \frac{1}{k!} \le \frac{1}{(n+1)!}\sum_{k=0}^{m-1} \frac{1}{n^k} \le \frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n}} < \frac{2}{(n+1)!}$, which is arbitrarily small for m and n large.

2.1.4; The functions $\{\sin n\pi x\}_{n=1}^{\infty}$ are mutually orthogonal and hence linearly independent.

2.1.5; $\max_t |f_n(t) - f_m(t)| = \frac{1}{2}(1 - \frac{n}{m})$ if $m > n$, which is not uniformly small for m and n large. However, $\int_0^1 |f_n(t) - f_m(t)|^2 dt = \frac{(n-m)^2}{12nm^2} < \frac{1}{12n}$.

2.1.11; $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = -\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = -\frac{\pi}{4}$. Fubini's theorem fails because $\int_0^1 \left(\int_0^1 |f(x,y)|dx\right) dy = \int_0^1 \left(\int_0^1 \frac{1}{x^2+y^2}dx\right) dy = \int_0^1 \frac{1}{y}\tan^{-1}\frac{1}{y}dy$ does not exist.

2.2.1; Using $w(x) = 1, p(x) = \frac{15}{16}x^2 + \frac{3}{16}$, with $w(x) = \sqrt{1-x^2}, p(x) = \frac{8}{15\pi}(6x^2+1)$, and with $w(x) = \frac{1}{\sqrt{1-x^2}}, p(x) = \frac{2}{3\pi}(4x^2+1)$.

2.2.2; With the additional assumption that $f(x) = \alpha + \beta x$ at two points $x = x_1$ and $x = x_2$, $x_1 < x_2$, we must have that $x_2 - x_1 = \frac{1}{2}$, and $x_2 + x_1 = 1$, so that $\alpha = \frac{3}{2}f(\frac{1}{4}) - \frac{1}{2}f(\frac{3}{4}), \beta = 2f(\frac{3}{4}) - 2f(\frac{1}{4})$.

2.2.3; (a) $g(x) = ax + bx^3 + cx^5$, where $a = \frac{105}{8\pi^5}(\pi^4 - 153\pi^2 + 1485) = 3.10346$, $b = -\frac{315}{4\pi^5}(\pi^4 - 125\pi^2 + 1155) = -4.814388$, $c = \frac{693}{8\pi^5}(\pi^4 - 105\pi^2 + 945) = 1.7269$.

(b) $g(x) = 3.074024x - 4.676347x^3 + 1.602323x^5$. A plot of $g(x)$ is barely distinguishable from $\sin \pi x$ on the interval $-1 \le x \le 1$.

2.2.4; Use integration by parts to show that the Fourier coefficients for the two representations are exactly the same for any function which is sufficiently smooth.

2.2.6; b) Write $\phi_{n+1} - A_n x \phi_n = \sum_{k=0}^{n} \beta_k \phi_k$, and evaluate the coefficients by taking inner products with ϕ_j, and using part a).

2.2.12; Direct substitution and evaluation of the x integral yields $h(t) = \sum_{k=-\infty}^{\infty} f_k g_k e^{ikt}$.

2.2.13; Use direct substitution and the fact that $\frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi ijk/N} = 1$ if k is an integer multiple of N (including 0), and $= 0$ otherwise.

2.2.14; Rewrite the definition of the discrete Fourier transform as a matrix multiplication. Show that the matrix is orthogonal.

2.2.16; $23 = 31 \oplus 15 \oplus 7$ where $\oplus$ means "add without carry".

2.2.17; (by induction) Suppose that $\mathrm{Wal}(n, 1-x) = (-1)^n \mathrm{Wal}(n, x)$. Then, if $0 < x < \frac{1}{2}$, $\mathrm{Wal}(2n, 1-x) = (-1)^n \mathrm{Wal}(n, 1-2x) = (-1)^{2n} \mathrm{Wal}(n, 2x) = \mathrm{Wal}(2n, x)$. Similarly, if $0 < x < \frac{1}{2}$, then $\mathrm{Wal}(2n+1, 1-x) = (-1)^{n+1} \mathrm{Wal}(n, 1-2x) = -\mathrm{Wal}(n, 2x) = \mathrm{Wal}(2n+1, x)$ as needed.

2.2.18; The relevant identites are:

(a) $2i \oplus 2j = 2(i \oplus j)$

(b) $2i \oplus (2j+1) = 2i \oplus 2(j-1) + 1 = 2(i \oplus (j-1) + 1) - 1$

(c) $(2i+1) \oplus (2j+1) = (2(i-1)+1) \oplus (2(j-1)+1) = 2(i-1) \oplus 2(j-1) = 2((i-1) \oplus (j-1))$

2.2.19; Hint: Find the binomial representation of x_j, given the binomial representation of j, and use this information in the representatio of $\mathrm{Wal}(k, x_j)$ and $\mathrm{Wal}(j, x_k)$.

2.2.24; (a) Differentiate the expresion $\int_0^1 \left(\sum_{i=0}^{N} f_i \phi_i(x) + \alpha_i \psi_i(x)\right)^2 dx$ with respect to α_j, set the derivative to zero, and reexpress these equations in matrix notation.

3.1.1; Use Leibniz rule to differentiate the expression $u(x) = \int_0^1 y(x-1)f(y)dy + \int_x^1 x(y-1)f(y)dy$ twice with respect to x'

3.2.1; Find a sequence of functions whose L^2 norm is uniformly bounded but whose value at zero is unbounded. There are plenty of examples.

3.2.2; The proof is the same for all bounded linear operators; see top of page 107.

3.2.3; The null space is spaned by $u = 1$ when $\lambda = 2$, therefore solutions exist and are unique if $\lambda \neq 2$, and solutions exist (but are not unique) if $\lambda = 2$ and $\int_0^{1/2} f(t)dt = 0$.

3.2.4; The null space is spanned by $u = x$ when $\lambda = 3$, therefore solutions exist and are unique if $\lambda \neq 3$, and solutions exist, but are not unique, if $\lambda = 3$ and $\int_0^1 tf(t)dt = 0$.

3.2.5; The null space is spanned by $\phi(x) = \cos jx$ if $\lambda = \frac{j}{\pi}$. Therefore, if $\lambda \neq \frac{j}{\pi}$ for $j = 1, \cdots, n$, the solution exists and is unique, while if $\lambda = \frac{j}{\pi}$ for some j, then a solution exists only if $\int_0^{2\pi} f(x) \cos jxdx = 0$.

3.3.1; $u(x) = f(x) + \lambda \int_0^{2\pi} \sum_{J=1}^n \frac{1}{j-n\pi} \cos jt \cos jx f(t)dt = \sin^2 x - \frac{\lambda}{2} \frac{\pi}{2-\lambda\pi} \cos 2x$, provided $\lambda \neq \frac{2}{\pi}$. For $\lambda = \frac{2}{\pi}$, the least squares solution is $u(x) = \frac{1}{2}$.

3.3.2; $u(x) = \frac{1}{3-6\lambda} P_0(x) + \frac{3}{3-2\lambda} P_1(x) + \frac{10}{15-6\lambda} P_2(x)$, provided $\lambda \neq \frac{3}{2}, \frac{5}{2}$.

Remark: It is helpful to observe that $x^2 + x = \frac{1}{3} P_0(x) + P_1(x) + \frac{2}{3} P_2(x)$.

3.4.1; (a) Eigenfunctions are $\phi_n(x) = \sin n\pi x$ for $\lambda_n = \frac{1}{n^2\pi^2}$.

3.4.2; (a) $\phi_1(x) = \sin x, \lambda_1 = \frac{\pi}{2}, \phi_2(x) = \cos x, \lambda_2 = \frac{\alpha\pi}{2}$.

(b) $\phi_n(x) = \sin nx, \lambda_n = \frac{\pi}{2(n-1)^2}$, for $n \geq 2$.

(c) There are no eigenvalues or eigenfunctions.

(d) $\phi_n(x) = \sin a_n x, \lambda_n = \frac{1}{a_n^2}$, where $a_n = \frac{(2n+1)\pi}{2}$.

3.4.3; $\lambda_n = \frac{8}{n^2\pi^2}$ is a double eigenvalue with $\phi_n(x) = \sin \frac{n\pi x}{2}, \psi_n(x) = \cos \frac{n\pi x}{2}$ for n odd.

3.5.1; $u(x) = f(x) + \int_0^x e^{x-t} f(t)dt = e^x$ when $f(x) = 1$.

3.5.2; $u(x) = f(x) + \int_0^x \sin(t - x) f(t)dt = \cos x$ when $f(x) = 1$.

3.5.3; $u(x) = f(x) + \int_0^x \sin(x - t) f(t)dt = e^x$ when $f(x) = 1 + x$.

3.5.4; $u(x) = f(x) + \frac{2\lambda}{2-\lambda} \int_0^{1/2} f(t)dt = x + \frac{\lambda}{4(2-\lambda)}$ when $f(x) = x$, provided $\lambda \neq 2$.

3.5.5; $u(x) = f(x) + \frac{3}{5} \int_0^1 xtf(t)dt = x$ when $f(x) = \frac{5x}{6}$.

4.1.1; (a) Use that $S_k(x) = \frac{1}{2}\frac{\sin(k+\frac{1}{2})\pi x}{\sin\frac{\pi x}{2}}$ for $-1 < x < 1$, and then observe that $\frac{\sin\frac{\pi x}{2}}{\frac{\pi x}{2}}S_k(x)$ is a delta sequence according to the text.

4.1.4; Observe that $\chi = \int_{-\infty}^{x}\psi(x)dx$ is a test function, $\chi(0) = 0$, so that $\chi = x\phi$ for some test function ϕ. Hence, $\psi(x) = \frac{d}{dx}(x\phi(x))$.

4.1.5; $u(x) = c_1 + c_2H(x) + c_3\delta(x)$.

Hint: Show that a test function ψ is of the form $\psi = \frac{d}{dx}(x^2\phi)$ if and only if $\int_{-\infty}^{\infty}\psi dx = \int_0^{\infty}\psi dx = \psi(0) = 0$.

4.1.7; Hint: Set $u = xv$, so that $x^2v' = 0$, and then $u(x) = c_1x + c_2xH(x)$ (using that $x\delta(x) = 0$).

4.1.8; $u(x) = c_1 + c_2H(x)$.

4.1.11; In the sense of distribution, $\chi'(x) = \delta(x) - \delta(x-1)$, since $\langle\chi'(x)\phi(x)\rangle = -\langle\chi(x), \phi'(x)\rangle = -\int_{\infty}^{\infty}\chi(x)\phi'(x)dx = -\int_0^1\phi'(x)dx = \phi(0) - \phi(1)$.

4.1.12; (a) For distributions f and g, deine $\langle f * g, \phi\rangle = \langle g, \psi$ $rangle$, where $\psi(t) = \langle f(x), \phi(x+t)\rangle$.

(b) $\delta * \delta = \delta(x)$.

4.2.1; $g(x,t) = -x$ for $0 \le x \le t, g(x,t) = g(t,x)$.

4.2.2; $U(x) = x$ is a solution of the homogeneous problem. There is no Green's function.

4.2.3; $g(x,t) = \frac{\cos\alpha(\frac{1}{2}-|x-t|)}{\sin\frac{\alpha}{2}}$ provided $\alpha \ne 2n\pi$.

4.2.4; $g(x,t) = (2t - t^2 - 1)x$ for $0 \le x < t \le 1$, and $g(x,t) = g(t,x)$.

4.2.5; $u = 1$ satisfies the homogeneous problem. There is no Green's function.

4.2.6; $g(x,t) = \begin{cases} -\frac{x}{5}(3t^{5/2}+2)\text{for}0 \le x < t \\ -\frac{t^{3/2}}{5}(3x+2x^{-3/2})\text{for}x > t \end{cases}$

4.2.9; $u(x) = \int_0^1 g(x,t)f(t)dt - \lambda\int_0^1 g(x,t)dt + \alpha(1-x) + \beta x$, where $g(x,t) = x(t-1)$ for $0 \le x < t \le 1, g(x,t) = g(t,x)$.

4.2.10; $u(x) = \int_0^1 g(x,t)f(t)dt - \lambda \int_0^1 g(x,t)dt$ where $g(x,t) = \frac{1}{3}(x+1)(t-2)$ for $0 \le x < t$, and $g(x,t) = g(t,x)$.

4.2.11; $u(x) = \int_0^1 g(x,t)f(t)dt - \lambda \int_0^1 g(x,t)dt$ where $g(x,t) = \frac{1}{2n}x^n(t^n - t^{-n})$ for $0 \le x < t \le 1, g(x,t) = g(t,x)$.

4.2.12; $g(x,t) = -\frac{1}{2}e^{-|t-x|}$, and $u(x) = \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-|t-x|}u(t)dt - \int_{-\infty}^{\infty} \frac{1}{2}e^{-|t-x|}f(t)dt$.

4.3.2; $L^*v = -(p(x)v')' + q(x)v$ with $p(0)v(0) = p(1)v(1)$, and $p(0)v'(0) = p(1)v'(1)$.$\hat{L}u = -(pu')' + qu - p(1)(u'(0) - u'(1))\delta(x-1) + p(1)(u(1) - u(0))\delta'(x-1)$.

4.3.3; $L^*v = v'' - 4v' - 3v$ with $v'(0) = 0, v'(1) = 0$.$\hat{L}u = u'' + 4u' - 3u + (4u(0) + u'(0))\delta(x) - (4u(1) + u'(1))\delta(x-1)$.

4.3.4; (a) $g^*(x,t)w(x) = g(t,x)w(t)$

(b) $u(t) = \int_a^b g(t,x)\hat{f}(x)dx$.

4.3.6; This follows from 4.3.2.

4.3.9; Require $\int_0^{2\pi} f(x)\sin x dx = \alpha$, and $\int_0^{2\pi} f(x)\cos x dx = \beta$.

4.3.10; Require $\int_0^1 f(x)dx = -\beta$.

4.3.11; Require $\int_0^{\frac{1}{2}} f(x)\sin \pi x dx = \beta + \pi\alpha$.

4.3.12; Require $\int_0^1 f(x)dx = \alpha - \beta$.

4.4.1; $g(x,t) = tx + \frac{1}{2}(t-x) + \frac{1}{12} - \frac{1}{2}(x^2 + t^2)$, for $0 \le x < t \le 1, g(x,t) = g(t,x)$.

4.4.2; $g(x,t) = (\frac{1}{2} + t^2(4t-3))(x - \frac{1}{2}) + \frac{1}{4} - \frac{t}{2} - \frac{x^2}{2} - xH(t-x)$.

4.4.3; $g(x,t) = -\frac{1}{8\pi^2}\cos 2\pi(x-t) - \frac{x-t}{2\pi}\sin 2\pi(x-t) - \frac{1}{4\pi}\sin 2\pi(t-x)$.

4.4.4; $g(x,t) = \frac{2x}{\pi}\cos x \sin t + \frac{2t}{\pi}\sin x \cos t - \frac{1}{\pi}\sin x \sin t - 2H(t-x)\sin x \cos t - 2H(x-t)\cos x \sin t$.

4.4.5; $g(x,t) = \frac{9}{5}xt - x - \frac{xt}{2}(x^2 + t^2)$ for $x < t, g(x,t) = g(t,x)$.

4.4.6; $g(x,t) = \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln 91 + t) + \frac{1}{2}$ for $-1 \le x < t \le 1, g(x,t) = g(t,x)$.

4.4.7; $u(x) = \frac{1}{8}\cos 2x + (\beta - \alpha)\frac{x^2}{2\pi} + \alpha x - \frac{\pi^2}{3}(\alpha + \frac{\beta}{2})$.

4.4.8; $u(x) = -\frac{3}{\pi}x\cos x + \cos x + \frac{1}{32}\sin 3x - \frac{3}{2\pi}\sin x$.

4.4.9; $u(x) = 0$.

4.5.1; $u(x) = \frac{\alpha-\beta}{6} - \frac{\alpha}{2} + \alpha x + \frac{\beta-\alpha}{2}x^2 + \sum_{n=1}^{\infty}\frac{b_n}{n^2\pi^2}\cos n\pi x$, where $b_n = -2\int_0^1 (f(x) + \alpha - \beta)\cos n\pi x dx$.

4.5.4; No eigenfunction expansion solution exists.

4.5.5; $u(x) = \alpha x + \beta - \alpha\pi + \sum_{n=1}^{\infty} a_n \cos(2n-1)\frac{x}{2}$, where $a_n = -\frac{8}{\pi(2n-1)^2}\frac{1-\cos(2n-1)\frac{\pi}{2}}{(2n-1)^2/2-1}$.

4.5.6; $u(x) = -\frac{c}{9}(\frac{3}{2}x^2 - \frac{b}{2}x$. Solution is exact if $a + \frac{c}{3} = 0$.

4.5.7; $u(x) = -cL_2(x) + (b+4c)L_1(x)$, where $L_1(x) = 1 - x$, and $L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$ are Laguerre polynomials.

4.5.8; (Use Hermite polynomials) $U9x) = (-\frac{c}{5}x - \frac{bx}{3} - a - \frac{4c}{5})e^{x^2/2}$.

4.5.9; Eigenfunctions are $\phi_n(x) = \sin\frac{n\pi}{2}(x+1), \lambda = \frac{n^2\pi^2}{4}$, so $a_n = 0$ for n even, $a_n = -\frac{1}{\pi}\frac{4n}{n^2-4}$ for n odd.

4.5.12; For $\lambda = 4\pi^2$, eigenfunctions are $1, \cos 2\pi x, \sin 2\pi x$. For $\lambda = 4\pi^2 n^2$ with $n > 1$, eigenfunctions are $\cos 2\pi nx$ and $\sin 2\pi nx$.

5.1.1; a) $y = \text{constant}$.

5.1.2; $y(x) = \frac{1}{2}(x^2 - 3x + 1)$

5.1.10; The Euler-Lagrange equation is $\frac{d^4y}{dx^4} - y = 0$.

5.2.2; $u_{xx} = 0$ and $\mu_1 u_{xxx} - \mu_2 u_x = 0$ at $x = 0, 1$.

5.2.3; If u is the vertical displacement of the string, require $\rho u_{tt} = \mu_1 \frac{\partial}{\partial x}\frac{u_x}{\sqrt{1+u_x^2}}$ subject to the boundary conditions $mu_{tt} + ku = \mu_1\frac{u_x}{\sqrt{1+u_x^2}}$ at $x = 0$, and $mu_{tt} + ku = -\mu_1\frac{u_x}{\sqrt{1+u_x^2}}$ at $x = 1$.

6.1.1; a) $f(-3) = -i\sqrt{84}$, $f(\frac{1}{2}) = -\sqrt{\frac{7}{8}}$, $f(5) = -\sqrt{20}$.

6.1.3; (a) $z = \frac{\pi}{2} + 2n\pi - \ln(2 \pm \sqrt{3})$

(b) $z = 2n\pi + i\ln(\sqrt{2}+1), z = (2n+1)\pi + i\ln(\sqrt{2}-1)$.

6.1.4; $i^i = e^{-(n/2+2n\pi)}$ for all integer n, $\ln(1+i)^{i\pi} = -\pi^2(\frac{1}{4}+2n) + \frac{i\pi}{2}\ln 2$, $\arctan 1$ has no value.

6.1.7; The two regions are $|z| < 1$ and $|z| > 1$; There are branch points at $w = \pm 1$.

6.2.1; $f(z) = \frac{15-8i}{4(z-2)^2(z-\frac{i}{2})}$.

6.2.2; $\int_C f(z)dz = -2\pi\sqrt{19}(15)^{1/3}e^{-i\pi/3}$

6.2.4; Use that $f(z) = z^{1/2}$ is an analytic function.

6.2.6; $\int_{|z|=1/2} \frac{z+1}{z^2+z+1}dz = 0$

6.2.7; $\int_{|z|=1/2} \exp[z^2\ln(1+z)]dz = 0$. (There is a branch point at $z = -1$.)

6.2.8; $\int_{|z|=1/2} \arcsin z dz = 0$ (There are branch points at $z = \pm 1$.)

6.2.9; $\int_{|z|=1} \frac{\sin z}{2z+i}dz = \pi \sinh \frac{1}{2}$

6.2.10; $\int_{|z|=1} \frac{\ln(z+2)}{z+2}dz = 0$

6.2.11; $\int_{|z|=1} \cot z dz = 2\pi i$

6.2.13; Hint: Use the transformation $z = \xi^\rho$ where $\rho = \frac{1}{\alpha}$ and apply the Phragmen-Lindelof theorem to $g(\xi) = f(z)$.

6.3.5; $\phi + i\psi = a - \frac{2}{3}(b-a) - \frac{4i(b-a)}{3\pi}\ln(\frac{z-1}{z+1})$

6.3.6; $\phi + i\psi = \frac{4(b-a)}{3\pi}\ln(\frac{z-1}{z+1}) - i\frac{2}{3}(b-a)$

6.4.1; $\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \frac{2\pi}{\sqrt{4ac-b^2}}$

6.4.2; $\int_0^{\infty} \frac{x\sin x}{a^2+x^2}dx = \frac{\pi}{2}e^{-|a|}$

6.4.3; $\int_0^{\infty} \frac{dx}{1+x^k} = \frac{\pi}{k\sin\frac{\pi}{k}}$

6.4.4; $\int_0^{\infty} \frac{dx}{(x+1)x^p} = \frac{\pi}{\sin \pi p}$

6.4.5; $\int_1^\infty \frac{x}{(x^2+4)\sqrt{x^2-1}}dx = \frac{\pi}{2\sqrt{5}}$

6.4.6; $\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi}{\sqrt{2}}$

6.4.8; $\int_{-\pi}^{\pi} \frac{x\sin x}{a^2-2a\cos x+1}dx = \frac{2\pi i}{a}(H(a-1)\ln a - \ln(a+1))$

6.4.9; $\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\cosh x}dx = \frac{\pi}{\cosh\frac{\omega\pi}{2}}$

6.4.11; $\int_0^\infty \frac{dx}{x^2+x+2} = \frac{1}{4}\ln\sqrt{2} - \frac{3}{4\sqrt{7}}(\arctan\sqrt{7}-\pi)$

6.4.12; $\int_0^{2\pi} \ln(a+b\cos\theta)d\theta = 2\pi\ln(\frac{a+\sqrt{a^2-b^2}}{2})$. Hint: Differentiate the integral with respect to b, and evaluate the derivative.)

6.4.13; $\int_0^\infty \frac{\sin\alpha x}{\sinh\pi x}dx = \frac{1}{2}\tanh\frac{\alpha}{2}$

6.4.14; $\int_0^\pi \ln(\sin x)dx = -\pi\ln 2$

6.4.15; $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}dx = \pi\ln 2$

6.4.20; $\int_0^\infty t^{-1/2}e^{i\mu t}dt = \sqrt{\frac{\pi}{|\mu|}}\exp(i\frac{\pi}{4}\sqrt{\frac{\mu}{|\mu|}})$

6.5.4; $f(y) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dy}\int_0^y \frac{T(x)}{(y-x)^{1-\alpha}}dx$. Hint: One of the ways to solve this problem is to use Laplace transforms and the convolution theorem.

6.5.6; $W(J_\nu, Y_\nu) = \frac{2}{\pi z}$

6.5.9 $Y_n(z) = \frac{2^n}{n\pi}\frac{\Gamma(n+1)}{z^n} +$ higher order terms.

6.5.11; Use that $\sum_{n=-\infty}^{\infty} J_n(z)t^n = e^{(t-1/t)z/2}$

6.5.26; Period $= \sqrt{\frac{21}{g}}B(\frac{1}{2},\frac{1}{4}) = \sqrt{\frac{21}{g}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$

7.1.2; (a) λ with $|\lambda| < 1$ is residual spectrum, with $|\lambda| = 1$ is continuous spectrum, and with $|\lambda| > 1$ is resolvent spectrum.

(b) Note that $L_2 = L_1^*$. λ with $|\lambda| < 1$ is residual spectrum, with $|\lambda| = 1$ is continuous spectrum, and with $|\lambda| > 1$ is resolvent spectrum.

(c) $\lambda_n = \frac{1}{n}$ for positive integers n are point spectrum, there is no residual spectrum since L_3 is self adjoint, and $\lambda \neq \frac{1}{n}$ is resolvent spectrum, $\lambda = 0$ is continuous spectrum.

7.1.3; Hint: Show that $\{x_n\}$ with $x_n = \sin n\theta$ is an improper eigenfunction.

7.1.4; Show that $\phi(x) = \sin \mu x$ is an eigenfunction for all μ. Notice that the operator is not self-adjoint.

7.2.1; $\delta(x-\xi) = 2\sum_{n=1}^{\infty} \sin(\frac{2n-1}{2}\pi x)\sin(\frac{2n-1}{2}\pi\xi)$

7.2.2; $\delta(x-\xi) = \frac{2}{\pi}\int_0^\infty \cos kx \cos k\xi dk$

7.2.3; $\delta(x-\xi) = \frac{2}{\pi}\int_0^\infty \sin k(x+\phi)\sin k(\xi+\phi)dk$ where $\tan\phi = \frac{k}{\alpha}$.

7.2.7; (a) $-i\mu F(\mu)$

(b) $-\frac{F(\mu)}{i\mu}$

(c) $e^{i\mu k}F(\mu)$

(d) $F(\mu+k)$

7.2.8; $\int_x^\infty e^{x-s}f'(s)ds$

7.2.9; Use the convolution theorem to find $u(x) = f(x) - \frac{4}{3}\int_{-\infty}^{\infty} f(t)e^{-3|x-t|}dt$.

7.3.3; (a) $u(x) = \int_0^x f(y)K(x-y)dy$ where $K = L^{-1}(\frac{1}{1+L(k(x))})$.

(b) $f(t) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dt}\int_0^t \tau^{\alpha-1}T(t-\tau)dt$

7.3.5; $M = \frac{1}{s}$

7.3.6; $M[(1+x)^{-1}] = \frac{\pi}{\sin \pi s}$.

7.3.7; $M[e^{-x}] = \Gamma(s)$

7.3.8; $M = \frac{1}{1-s}$

7.3.9; $M[e^{ix}] = i^s\Gamma(s)$

7.3.10; $M[\cos x] = \frac{1}{2}\cos\pi s\Gamma(s)$

7.3.12; $F(\mu) = \int_0^\infty rf(r)\sin\mu r dr, rf(r) = \frac{2}{\pi}\int_0^\infty F(\mu)\sin\mu r d\mu$.

7.4.2; Let $u_n = \sum_j g_{nj}f_j$ where $g_{nj} = 0, n \le j, g_{nj} = \frac{\mu}{\mu^2-1}(\mu^{n-j}-\mu^{j-n})$ where $\mu^2 - \lambda\mu + 1 = 0$.

7.5.1; $u_1(x) = \begin{cases} \cos x, x > 0 \\ \cosh x, x < 0 \end{cases}$ $u_2(x) = \begin{cases} \sin x, x > 0 \\ \sinh x, x < 0 \end{cases}$

7.5.3; Eigenvalues are $\lambda = -\mu^2$ where $\tanh \mu = -\frac{\mu}{A+\mu}, \mu > 0$.

7.5.4; $\lambda^2 = A - \mu$ where $\tan a\sqrt{A-\mu} = \frac{\sqrt{A-\mu}}{\mu}$.

7.5.9; $R = -e^{2ik_1 a} \frac{k_2 \cos k_2 a + ik_1 \sin k_2 a}{k_2 \cos k_2 a - ik_1 \sin k_2 a}$ where $k_i = \frac{\omega}{c_i}$.

7.5.9; $T_r(k) = \frac{(ik-2)(ik-1)}{(ik+2)(ik+1)}$

7.5.15; $R = -e^{-2ika} \frac{(ik+\tanh a)(ik-1)}{(ik-\tanh a)(ik+1)}$. There is one bound state at $k = -i \tanh a$ if $a < 0$ having $u(x) = (\tanh a - \tanh x)e^{\tanh a(x-a)}$.

8.1.3; (a) Require $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\phi_t + c\phi_x) dx dt = 0$ for all test functions $\phi(x,t)$.

(b) If $u = f(x - ct)$, make the change of variables $\xi = x + ct, \eta = x - ct$ to find $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\phi_t + c\phi_x) dx dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi)\phi_\eta d\xi d\eta = 0$ since $\int_{-\infty}^{\infty} \phi_\eta d\eta = 0$.

8.1.4; $G(z, z_0) = -\frac{1}{2\pi} \ln |\frac{z-z_0}{zz_0-1}|$.

8.1.7; (a) Using the Fourier transform in x, $G(x, y, x_0, y_0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ik(x-x_0)) \frac{\cosh k(y_0-a) \sinh ky}{a \cosh ka} dk$ for $y > y_0$.

(b) Using Fourier series in y, $G(x, y, x_0, y_0) = \sum_{n=1}^{\infty} \frac{1}{a\lambda_n} \exp(-\lambda_n |x-x_0|) \sin \lambda_n y \sin \lambda_n y_0$ where $\lambda_n = \frac{2n+1}{2} \frac{\pi}{a}$.

8.1.9; Suppose $f(\theta) = \sum_{n=0}^{\infty} a_n \cos n(\theta - \phi_n)$ then $u(r, \theta) = \sum_{n=0}^{\infty} a_n (\frac{a}{r})^n \cos n(\theta - \phi_n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta + \frac{1}{\pi} \sum_{n=1}^{\infty} (\frac{a}{r})^n \int_0^{2\pi} f(\phi) \cos n(\theta - \phi) d\phi$. This infinite sum can be summed by converting the cosine to complex exponentials and using geometric series.

8.1.14; Using Mellin transforms $u(r, \theta) = \frac{1}{2\pi} \int_0^{\infty} e^{-ik \ln r} \frac{G(k) \sinh k\theta - kF(k) \cosh k(\theta - \pi)}{k \cosh k\pi} dk$ where $F(k) = \int_0^{\infty} e^{ik \ln r} \frac{f(r)}{r} dr, G(k) = \int_0^{\infty} e^{ik \ln r} \frac{g(-r)}{r} dr$.

8.1.17; Eigenfunctions are $J_n(\mu_{nk} \frac{r}{R}) \sin n\theta$ for $n > 0$. Thus, eigenvalues are the same as for the full circle, with $n = 0$ excluded.

8.1.19; Eigenfunctions are $\phi_{nm}(\phi, \theta) = P_m^n(\cos \theta) \sin n\phi$ (or $\cos n\phi$) with $\lambda_{nm} = \frac{1}{R}\sqrt{m(m+1)}$.

8.1.22; Eigenfunctions are $u(r,\theta,\phi) = \frac{1}{\sqrt{r}} J_{m+1/2}(\mu_{mk}\frac{r}{R}) P_m^n(\cos\theta)\cos n\phi$ with eigenvalues $\lambda_{mk} = (\frac{\mu_{mk}}{R})^2$, where $J_{m+1/2}(\mu_{mk}) = 0$ for $m \geq n$. Note that $\mu_{01} = \pi, \mu_{11} = 4.493, \mu_{21} = 5.763, \mu_{02} = 2\pi, \mu_{31} = 6.988$, etc.

8.2.6; Hint: Compare the relative amplitudes of the harmonics in the two cases.

8.2.7; For a rectangle with sides a and b, $\omega = \frac{\lambda}{2\pi} = \frac{1}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$. Set $A = ab$, and find that the minimum is at $a = \sqrt{A}$.

8.2.8; For a square of side L, the fundamental eigenvalue is $\lambda = \sqrt{2}\frac{\pi}{L}$, whereas for a circle of radius R the fundamental eigenvalue is $\lambda = \frac{2.40482}{R}$. Take $\pi R^2 = L^2$ and use that $\frac{2\pi\omega}{c} = \lambda$.

8.2.11; Construct the Green's function from $H_0^{(1)}(\lambda|r-\xi|)$ with $\lambda = \frac{\omega}{c}$, and then the solution is proportional to (up to a scalar constant) $\psi(r,\theta) = \frac{1}{\sqrt{r}} e^{i\lambda r} \frac{\sin(\lambda a \sin\theta)}{\lambda \sin\theta}$ for large r.

8.3.7; $x = \ln 2\sqrt{\frac{2D}{\omega}} = 0.82\text{m}$, $t = \frac{\ln 2}{\omega} = 3.47 \times 10^6\text{s} = 40$ days.

8.4.1; $u_n(t) = \exp(-(\frac{2\sin(\pi/k)}{h})^2)\sin(\frac{2n\pi}{k})$. If we set $n = \frac{kx}{L}$, and $h = \frac{1}{k}$, we have in the limit $k \to \infty$, $u(x,t) = \exp(-\frac{4\pi^2}{l^2})\sin(\frac{2\pi x}{L})$, which is the correct solution of the continuous heat equation with periodic initial data.

8.4.2; $u_n(t) = J_n(-\frac{t}{h})$

8.4.5; $u_n(t) = J_{2n}(\frac{2t}{h})$

8.4.6; $k(\omega) = \cos^{-1}(1 - \frac{\omega^2 h^2}{2})$

9.4.5; $a_n W_n = \frac{\alpha(z)}{2}(z - \frac{1}{z})$

10.2.1; $E_n(x) = \frac{e^{-x}}{\Gamma(n)} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n+k)}{x^{k+1}}$.

10.2.3; $\int_0^1 e^{ixt} t^{-1/2} dt = \sqrt{\frac{\pi i}{x}} + \frac{e^{ix}}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^{k-1} \frac{\Gamma(k+1/2)}{x^{k+1}}$.

10.3.1; $E_1(x) = e^{-x} \sum_{k=0}^{\infty} (-1)^k \frac{k!}{x^{k+1}}$.

10.3.2; $\int_0^{\infty} \frac{e^{-zt}}{1+t^4} dt = \sum_{k=0}^{\infty} (-1)^k \frac{(4k)!}{z^{4k+1}}$.

10.3.2; $\int_0^1 e^{-xt}t^{-1/2}dt = \sqrt{\frac{\pi}{x}} - e^{-x}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{1}{x^{k+1}\Gamma(1/2-k)}$.

10.3.6; $\int_0^{\infty} e^{xt}t^{-t}dt = \sqrt{2\pi y}e^{y}(1 - \frac{1}{24y} - \frac{23}{576y^2} + \cdots)$, where $y = e^{x-1}$.

10.3.8; $\int_0^{\pi} e^{xt^2}t^{-1/3}\cos t dt = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{\Gamma(k+1/3)}{x^{k+1/3}}$.

10.3.11; b) $\sum_{k=0}^{n}\binom{n}{k}k!n^{-k} = \sqrt{\frac{n\pi}{2}}$ to leading order for large n.

10.3.13; $\int_0^{\infty} t^x e^{-t}\ln t dt = \sqrt{2\pi}x^{x+1}e^{-x}\ln x(\frac{1}{2x^{3/2}} - \frac{1}{24x^{5/2}} + \cdots)$.

10.4.1; $\int_C \frac{e^{k(z^2-1)}}{z-1/2}dx = 2\pi i e^{-3k/4} - 2ie^{-k}\sum_{j=0}^{\infty}(-4)^j\frac{\Gamma(j+1/2)}{k^{j+1/2}}$.

10.4.3; $I(x) = \sqrt{\frac{\pi}{x}}e^{-2x/3}(1 - \frac{5}{48x} + \frac{385}{4608x^2} + O(x^{-3}))$

10.4.4; $J_n(z) = \sqrt{\frac{2}{z\pi}}\cos(z - \frac{n\pi}{2} - \frac{\pi}{4}) - \frac{4n^2-1}{8}\sqrt{\frac{2}{z^3\pi}}\sin(z - \frac{n\pi}{2} - \frac{\pi}{4}) + O(z^{-5/2})$

10.4.6; $\int_0^1 \cos(xt^p)dt = \frac{1}{p}(\frac{i}{x})^{1/p}\Gamma\frac{1}{p}) - \frac{ie^{ix}}{px}$.

10.4.10; $J_n(\lambda_k) = 0$ for $\lambda_k = K - \frac{4n^2-1}{8K} + O(K^{-2})$ where $K = (2k+n+3)\frac{\pi}{2}$.

10.5.1; $\int_a^b f(x)e^{ikg(x)}dx = f(\alpha)\Gamma(\frac{4}{3})(\frac{kg^{(3)}(\alpha)}{6})^{2/3}e^{ikg(\alpha)}$.

12.1.3; With $\tau = \epsilon^2 t, \mu = \epsilon^3\lambda$ the Landau equation is $A_\tau = \frac{1}{2}A(\lambda - A^2)$.

12.1.4; $u(t) = \frac{A_0}{\sqrt{1+\frac{3}{4}A_0\epsilon t}}\sin(t+\phi_0) + O(\epsilon)$

12.2.4; $u(t) = \frac{1}{1+t} + O(\epsilon), v(t) = \frac{-1}{(1+t)^2} + e^{-t/\epsilon} + O(\epsilon)$

12.3.3; $u(t) = -\tan^{-1}(t) + \epsilon^{1/2}\exp(\frac{-t}{2\epsilon^{1/3}})\left(\frac{1}{3}\sin(\frac{\sqrt{3}t}{2\epsilon^{1/3}}) - \cos(\frac{\sqrt{3}t}{2\epsilon^{1/3}})\right) + \frac{1}{4}\epsilon^{1/3}2^{2/3}\exp(\frac{2^{1/3}(t-1)}{\epsilon^{1/3}})$

12.3.7; (a) $u(x) = \frac{x}{x+1} - \tanh(\frac{x}{2\epsilon} - \tanh^{-1}(\frac{2}{3}))$.

(b) $u(x) = 4\frac{x-1}{2x-3} - 2\tanh(\frac{x-1}{\epsilon} + \tanh^{-1}(\frac{1}{4}))$

(c) $u(x) = H(x-\frac{1}{4})\frac{4x-1}{5(x+1)} + \frac{2}{5}(1 - H(x-\frac{1}{4}))\frac{11-4x}{2x-3} - \frac{4}{5}\tanh(\frac{2}{5\epsilon}(x-\frac{1}{4}))$ where $H(x)$ is the usual Heaviside function.

12.3.8; $u(x) = \frac{\alpha+\beta-1}{2} + \epsilon\ln(\cosh(\frac{2t-\alpha+\beta-1}{2\epsilon})) + O(\epsilon)$