1.3.5; Suppose A is a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 > \lambda_3 \geq \ldots$. Show that

$$\max_{\langle u,v\rangle=0}\langle Au, u\rangle + \langle Av, v\rangle = \lambda_1 + \lambda_2$$

where $||u|| = ||v|| = 1$.

Use Lagrange multipliers and look for extremals of

$$P(u, v) = \langle Au, u\rangle + \langle Av, v\rangle + \mu_1(\langle u, u\rangle - 1) + \mu_2(\langle v, v\rangle - 1) + \mu_3|\langle u, v\rangle|^2$$

Differentiate $P(u, v)$ with respect to parameters μ_1, μ_2, and μ_3 and set the derivatives to zero to learn (as expected) that $||u||^2 = ||v||^2 = 1$, and $\langle u, v\rangle = 0$. Now set $u = u_0 + h$, $v = v_0 + g$, expand $P(u, v)$ and require that the linear terms vanish. That is, require

$$\text{Re } (\langle Au + \mu_1 u, h\rangle + \langle Av + \mu_2 v, g\rangle) = 0$$

for all g, h. It follows that we must have

$$Au + \mu_1 u = 0, \qquad Av + \mu_2 v = 0.$$

In other words, u and v must be eigenvectors for A.

Now suppose that the normalized eigenvectors of A are $\{\phi_i\}$ with corresponding eigenvalues λ_i. Since we know that u and v must be eigenvectors, we have that

$$\begin{aligned}
\max_{\langle u,v\rangle=0}\langle Au, u\rangle + \langle Av, v\rangle &= \max_{i\neq j}\langle A\phi_i, \phi_i\rangle + \langle A\phi_j, \phi_j\rangle \\
&= \max_{i\neq j}\lambda_i + \lambda_j \\
&= \lambda_1 + \lambda_2
\end{aligned}$$

∎

2.2.21; Verify that the wavelet generated by the sinc function is

$$W(x) = \text{sinc}(x - \frac{1}{2}) - 2\,\text{sinc}(2x - 1).$$

Recall that

$$\begin{aligned}
\text{sinc}(x) &= \sum_k \text{sinc}(\frac{k}{2})\,\text{sinc}(2x - k) \\
&= \text{sinc}(2x) + \sum_{k\text{odd}} \text{sinc}(\frac{k}{2})\,\text{sinc}(2x - k)
\end{aligned}$$

so that

$$\sum_{k\text{odd}} \operatorname{sinc}(\frac{k}{2}) \operatorname{sinc}(2x - k) = \operatorname{sinc}(x) - \operatorname{sinc}(2x)$$

It follows that

$$\begin{aligned}
W(x) &= \sum_k (-1)^k \operatorname{sinc}(\frac{k-1}{2}) \operatorname{sinc}(2x - k) \\
&= \sum_k (-1)^{k+1} \operatorname{sinc}(\frac{k}{2}) \operatorname{sinc}(2x - 1 - k) \\
&= -\operatorname{sinc}(2x - 1) + \sum_{k\text{odd}} \operatorname{sinc}(\frac{k}{2}) \operatorname{sinc}(2x - 1 - k) \\
&= -2 \operatorname{sinc}(2x - 1) + \operatorname{sinc}(x - \frac{1}{2}).
\end{aligned}$$

∎

8.3.15 (A new problem) Under what conditions is the interchange $\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \frac{\partial u}{\partial t} dx$ valid?

Answer: Use Fubini's theorem to show that the interchange is valid if $\frac{\partial u}{\partial t}$ is absolutely integrable.