Which Finite Groups Act Freely on Spheres?

Adam Keenan

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We all know from algebraic topology that $\mathbb{Z}_2$ is the only group that can act freely on even dimensional spheres. What can we say about a finite group that acts freely on an odd dimensional sphere?

Proposition 1. *$\mathbb{Z}_n$ is the only finite group that can act freely on S^1.*

Proof. Let G be a finite group that acts freely on S^1, then S^1/G is homeomorphic to S^1. We have that $G \cong \pi_1(S^1/G)/p_*(\pi_1(S^1))$. Therefore G is cyclic. □

Proposition 2. *The group $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is prime, cannot act freely on any sphere.*

Proof. Suppose that $\mathbb{Z}_p \times \mathbb{Z}_p$ acts freely on some sphere S^n. By the previous proposition we may assume that $n > 1$. We can make a $K(\mathbb{Z}_p \times \mathbb{Z}_p, 1)$ by attaching an $n+1$-cell and then higher dimensional cells, if neccesary, to $S^n/\mathbb{Z}_p \times \mathbb{Z}_p$. This $K(\mathbb{Z}_p \times \mathbb{Z}_p, 1)$ that we have constructed has $n+1$ cohomology equal to $\mathbb{Z}_p$ or 0, if we use cohomology with $\mathbb{Z}_p$ coefficients. But this contradicts the Künneth formula. □

For those who know about group cohomology will know that if a group acts freely on sphere, then it has periodic cohomology. Now the group $\mathbb{Z}_p \times \mathbb{Z}_p$ does not have periodic cohomology, (just use the Künneth formula again) therefore it cannot act freely on any sphere. For those who do not know about group cohomology a finite group having periodic cohomology is equivalent to all the abelian subgroups being cyclic. Is periodic cohomology a sufficient condition for a finite group to act freely on a sphere? For example, S_3 (the symmetric group on three elements) has periodic cohomology, so can it act freely on some sphere. We shall see that the answer to this question is no.

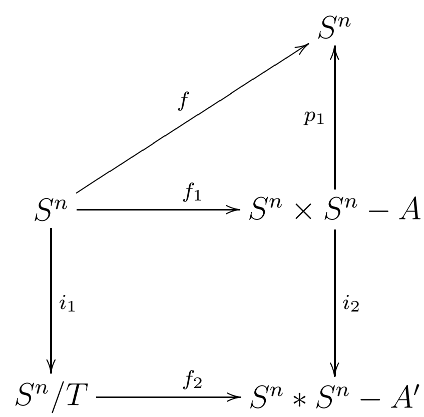
Definition 1. *The symmetric product of S^n with S^n denoted $S^n * S^n$ is the quotient space $S^n \times S^n / \sim$, where $(x, y) \sim (y, x)$.*

Example 1. *$S^1 * S^1$ is the Möbius band.*

Theorem 1 (Milnor-1957). *Let $T\colon S^n \longrightarrow S^n$ be a free involution. Then for every map $f\colon S^n \longrightarrow S^n$ of odd degree there exists a point $x \in S^n$ such that $Tf(x) = fT(x)$.*

As an exercise try to prove this theorem for the antipodal map.

Proof. Let A denote the set of all points (x, Tx) in $S^n \times S^n$. Let $S^n * S^n$ denote the symmetric product, and let A' denote the set of points $\{x, Tx\}$ in $S^n * S^n$. If $f\colon S^n \longrightarrow S^n$ satisfies $Tf(x) \neq fT(x)$ for all $x \in S^n$, then there is a commutative diagram



where i_1 and i_2 are the natural identification map, and where

$$f_1(x) = (f(x), f(Tx)), \quad f_2\{x, Tx\} = \{f(x), f(Tx)\}, \quad p_1(x, y) = x.$$

Thus we are almost able to factor f through "projective space" S^n/T, except that the map i_2 goes in the wrong direction.

Lemma 1. *The homomorphism*

$$(i_2)_*\colon H_k(S^n \times S^n - A) \longrightarrow H_k(S^n * S^n - A')$$

is an isomorphism for all k.

(Singular homology with the coefficient group $\mathbb{Z}_2$ is to be understood.)

Proof. We shall first show that the projection $p_1\colon S^n \times S^n - A \longrightarrow S^n$ is a locally trivial fibre map with fibre $p_1^{-1}(x) \simeq S^n - Tx$. Let $\overline{C}$ be any closed

n-cell in S^n with interior C, choose a homeomorphism g of $T\overline{C}$ onto the unit ball $\overline{B}$ in Euclidean n space. Define the map $\lambda\colon B \times \overline{B} \longrightarrow \overline{B}$ by

$$\lambda(v, w) = w + (1 - \|w\|)v,$$

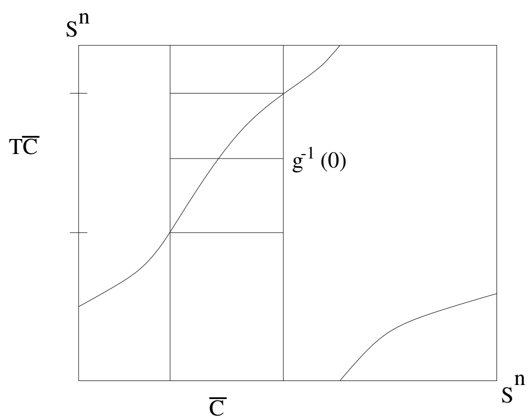
where B denotes the interior of $\overline{B}$. Note that for each fixed vector $v \in B$ the correspondence $w \longrightarrow \lambda(v, w)$ defines a homeomorphism of $\overline{B}$ onto itself leaving the boundary fixed and carrying the centre 0 into v. The required local product structure

$$\phi\colon C \times (S^n - g^{-1}(0)) \longrightarrow p_1^{-1}(C)$$

is now defined by

$$\phi(x, y) = (x, y) \text{ if } y \notin TC \quad \phi(x, y) = (x, g^{-1}\lambda(gTx, gy)) \text{ if } y \in T\overline{C}.$$

It is not hard to show that ϕ is a homeomorphism.



Thus p_1 is a fibre map. The fibres are contractible, therefore the induced homomorphism

$$(p_1)_*\colon H_*(S^n \times S^n - A) \longrightarrow H_*(S^n)$$

is an isomorphism. Let Δ denote the diagonal in $S^n \times S^n - A$ and let Δ' denote its image in $S^n * S^n - A'$. We have the following commutative diagram

$$\begin{array}{ccc} H_*(S^n \times S^n - A) & \xrightarrow{(p_1)_*} & H_*(S^n) \\ & {\scriptstyle i_*}\nwarrow \quad \swarrow{\scriptstyle d_*} & \\ & H_*(\Delta) & \end{array}$$

where $d\colon S^n \longrightarrow \Delta$ is the diagonal map and $i\colon \Delta \longrightarrow S^n \times S^n - A$ is the inclusion map. It follows that the inclusion homomorphism $H_*(\Delta) \longrightarrow H_*(S^n \times S^n - A)$ is an isomorphism. Hence the group $H_*(S^n \times S^n - A, \Delta)$ is zero. Next we shall show that the "relative" 2-fold covering map

$$(S^n \times S^n - A, \Delta) \longrightarrow (S^n * S^n - A', \Delta')$$

has a Gysin cohomology sequence. The difficulty lies in the fact that this map is a covering only in the complement of the diagonal Δ. If U denotes any symmetric neighbourhood of Δ and U' denotes its image in the collapsed space, consider the diagram

$$\begin{array}{ccccc}
(S^n \times S^n - A - \Delta, U - \Delta) & \xrightarrow{e} & (S^n \times S^n - A, U) & \longleftarrow & (S^n \times S^n - A, \Delta) \\
\big\downarrow{\scriptstyle c_1} & & \big\downarrow{\scriptstyle c_2} & & \big\downarrow{\scriptstyle c} \\
(S^n * S^n - A' - \Delta', U' - \Delta') & \xrightarrow{e'} & (S^n * S^n - A', U') & \longleftarrow & (S^n * S^n - A', \Delta').
\end{array}$$

Since c_1 is a true 2-fold covering, it has a Gysin sequence. Since e and e' are excisions, the map c_2 also has a Gysin sequence. But since Δ and $S^n \times S^n - A$ are both absolute neighbourhood retracts, the cohomology group $H^*(S^n \times S^n - A, \Delta)$ is the direct limit of the groups $H^*(S^n \times S^n - A, U)$ as U ranges over all the symmetric neighbourhoods. A similar description holds for $H^*(S^n * S^n - A', U')$. Since the direct limit of exact sequences is again exact (it is an exact functor), this gives us the required Gysin sequence

$$\ldots \to H^k(S^n \times S^n - A, \Delta) \to H^k(S^n * S^n - A', \Delta')$$
$$\to H^{k+1}(S^n * S^n - A', \Delta') \to H^{k+1}(S^n \times S^n - A, \Delta) \to \ldots$$

Since the right and left hand groups in this sequence are known to be zero, induction shows that the middle groups are also zero. Therefore the corresponding homology groups $H_k(S^n * S^n - A', \Delta')$ are zero. This implies that the inclusion $H_*(\Delta') \longrightarrow H_*(S^n * S^n - A')$ homomorphism is an isomorphism. It has already been shown that the inclusion homomorphism $H_*(\Delta) \longrightarrow H_*(S^n \times S^n - A)$ is an isomorphism. Since the map $i_2|_\Delta\colon \Delta \longrightarrow \Delta'$ is a homeomorphism, this implies that

$$(i_2)_*\colon H_k(S^n \times S^n - A) \longrightarrow H_k(S^n * S^n - A')$$

is an isomorphism; which completes the proof of lemma 1. □

Consider the diagram

$$\begin{array}{ccc}
 & & H_n(S^n) \\
 & \overset{f_*}{\nearrow} & \uparrow{\scriptstyle (p_1)_*} \\
H_n(S^n) & \xrightarrow{(f_1)_*} & H_n(S^n \times S^n - A) \\
\downarrow{\scriptstyle (i_1)_*} & & \downarrow{\scriptstyle (i_2)_*} \\
H_n(S^n/T) & \xrightarrow{(f_2)_*} & H_n(S^n * S^n - A')
\end{array}$$

We have $(i_1)_* = 0$. Since $(i_2)_*$ is an isomorphism, this implies that $(f_1)_* = 0$ and hence $f_* = 0$. This contradicts the fact that f is odd. □

Corollary 1. *Suppose that a finite group G acts freely on a sphere, then any element of order two is contained in the centre of G.*

Proof. Let $g \in G$ be an element of order two, and let h be any element in G. Then g and h induce homeomorphisms T and f. With T a free involution and f a map of odd order (it has order plus or minus 1). Then there exists a point $x \in S^n$ such that $Tf(x) = fT(x)$. Therefore g commutes with h. □

This shows that S_3 cannot act freely on any sphere, even though it has periodic cohomology. Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ cannot act freely on any sphere, this shows that if G is a finite group which acts freely on a sphere, then it has at most one element of order two. Therefore the dihedral groups cannot act freely on any sphere. So we have shown the following:

Theorem 2. *Let G be a finite group acting on a sphere, then*

(1) All the abelian subgroups of G are cyclic.

(2) All elements of order 2 are central, hence G has at most one element of order 2.

It turns out that if a finite group G satisfies (1) and (2) in the theorem, then it does in fact act on some sphere [2]. What about if we look at linear actions on spheres. In general a linear action can be constructed as follows: a subgroup G of $U(n)$ acts on the homogeneous space $S^{2n-1} \cong U(n)/U(n-1)$;

if no conjugate of G intersects $U(n-1)$ non-trivially, then this gives rise to a free linear action on the sphere. For the three sphere we have the following theorem:

Theorem 3. *The following is a list of all finite groups which act linearly on S^3 without fixed points.*

(1) The groups 1, D^*_{4n}, O^*, and I^*.

(2) The groups $D_{2^k(2n+1)}$ with presentation

$$< x, y \colon x^{2^k} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1} >, \text{ where } k \geq 2, n \geq 1.$$

(3) The groups $P'_{8\cdot 3^k}$ with presentation

$$< x, y, z \colon x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 >, \text{ where } k \geq 1.$$

(4) The direct product of any of these groups with a cyclic group of relatively prime order.

The proof of this requires representation theory and would take too long to prove in this talk. However, we shall prove the following:

Theorem 4. *Every finite subgroup of S^3 (the unit quarternions) is a cyclic, binary dihedral or binary polyhedral group. Moreover, if two finite subgroups of S^3 are isomorphic, then they are conjugate.*

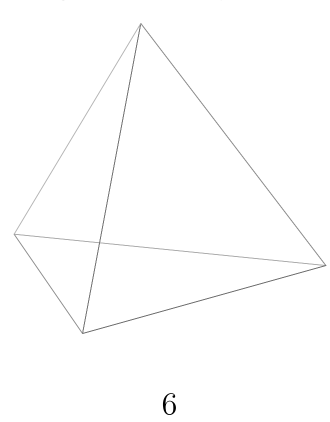
Before giving the proof we shall define the binary dihedral groups and the binary polyhedral groups.

The dihedral group D_{2n} is the group of symmetries of an n-gon and hence gives us an element in $SO(3)$.

Let T be the group of rotations of the tetrahedron. The presentation for T is given by

$$< a, b, c \colon a^3 = b^2 = c^2 = 1, bc = cb, aba^{-1} = c, aca^{-1} = bc > .$$

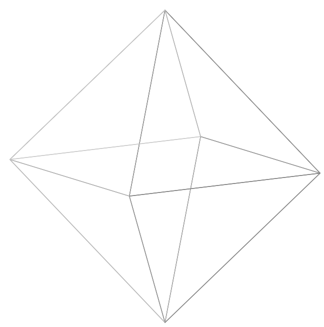
T is called the *tetrahedral group*. It has order 12 and is given as a subgroup of $SO(3)$. In terms of known groups it is just A_4.



Let O be the group of rotations of the octahedron. The presentation for O is given by

$$< a, b, c, d \colon a^3 = b^2 = c^2 = d^2 = 1, bc = cb, dad^{-1} = a^{-1}$$
$$aba^{-1} = c, aca^{-1} = bc, dbd^{-1} = cb, dcd^{-1} = c^{-1} > .$$

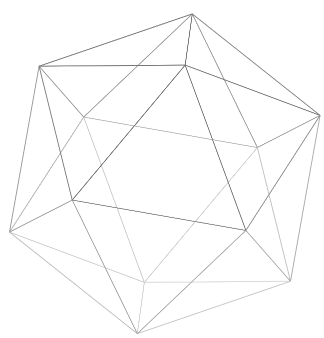
O is called the *octahedral group*. It has order 24 and is given as a subgroup of $SO(3)$. In terms of known groups it is just S_4.



Let I be the group of rotations of the icosahedron. The presentation for I is given by

$$< a, b, c \colon a^3 = b^2 = c^5 = abc = 1 > .$$

I is called the *icosahedral group*. It has order 60 and is obtained as a subgroup of $SO(3)$. In terms of known groups it is just A_5.



We have a double cover $\pi \colon S^3 \longrightarrow SO(3)$. The *binary dihedral* and the *binary polyhedral* groups are defined by

$$D^*_{2n} = \pi^{-1}(D_{2n}), \quad T^* = \pi^{-1}(T) \quad O^* = \pi^{-1}(O) \quad I^* = \pi^{-1}(I).$$

T^*, O^* and I^* are called the *binary tetrahedral*, *binary octahedral* and *binary icosahedral* groups, repectively. The binary icosahedral group is an example of a perfect group. The binary isosahedral group can be used to form a three manifold which has the same homology groups as the three sphere but is not homotopic to the three sphere.

Theorem 5. *Every finite subgroup of $SO(3)$ is a cyclic, dihedral, tetrahedral, octahedral or icosahedral group. If two finite subgroups of $SO(3)$ are isomorphic, then they are conjugate in $SO(3)$.*

Using this theorem we can now prove theorem 4.

Proof. Let H be a finite subgroup of S^3, and let $G = \pi(H)$. Suppose first that $H \neq \pi^{-1}(G)$. Then H is isomorphic to G and $\pi^{-1}(G) = H \times \{\pm 1\}$. As -1 is the only element of order 2 in S^3 it follows that H has odd order. Now G has odd order, so G is cyclic. Hence H is cyclic. Conversely, if H is a cyclic group of odd order in S^3, then H is isomorphic to $\pi(H)$ because $H \cap ker\pi = H \cap \{\pm 1\} = \{1\}$, so $H \neq \pi^{-1}(H)$. Now suppose that $H = \pi^{-1}(G)$. Then H has even order, containing the kernel of π. If G is cyclic of some order m, $G = \mathbb{Z}_m$, let g be a generator of G. Choose $f \in \pi^{-1}(g)$. Now $f^m = \pm 1$. If $f^m = -1$, then H is isomorphic to $\mathbb{Z}_{2m}$. If $f^m = 1$, then $H_1 =< f >$ has odd order, so m is odd and $H = H_1 \times \{\pm 1\} \cong \mathbb{Z}_{2m}$. If G is not cyclic, then, by definition, H is binary dihedral or binary polyhedral. Let H_1 and H_2 be finite isomorphic subgroups of S^3. If they have odd order, then $H_i \cong \pi(H_i) = G_i$. The G_i are conjugate in $SO(3)$, $G_1 = aG_2a^{-1}$. Choose $b \in \pi^{-1}(a)$; then $H_1 = bH_2b^{-1}$. If H_i have even order then $H_i = \pi^{-1}(G_i)$. G_1 and G_2 are isomorphic, G_i being the quotient of H_i by its unique subgroup of order 2. Now the G_i are conjugate and it follows that the H_i are conjugate.

□

If we let $\mathbb{Z}_2$ act on the sphere in the obvious way, then this action is linear. There does exist an action of $\mathbb{Z}_2$ on some sphere which is not linear. Also, there exists groups such that the only action they have on spheres is non-linear. We shall now take a look at what I hope to talk about next time.

Definition 2. *A faithful unitary representation $G \subset U(n)$ has* fixity *f if f is the smallest integer so that the induced action of G on $U(n)/U(n-f-1)$ is free.*

Theorem 6. *A subgroup $G \subset U(n)$ of fixity one acts freely and smoothly on $S^{2n-1} \times S^{4n-5}$.*

Corollary 2. *If G is any finite subgroup of $SU(3)$, then it will act freely and smoothly on $S^5 \times S^7$.*

For example, the groups A_5 and $Sl_3(\mathbb{F}_2)$ act freely on $S^5 \times S^7$. The big open problem in this field is the following:

Conjecture 1. *The group $(\mathbb{Z}_p)^{r+1}$ does not act freely on $S^{n_1} \times \cdots \times S^{n_r}$.*

The result is known for $r = 1$ and $r = 2$.

References

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[3] J. Wolf, *Spaces of Constant Curvature*, Publish or Perish, 1984.