From Coulomb to Stokes:
Potential Theory Applied to Fluid Mechanics

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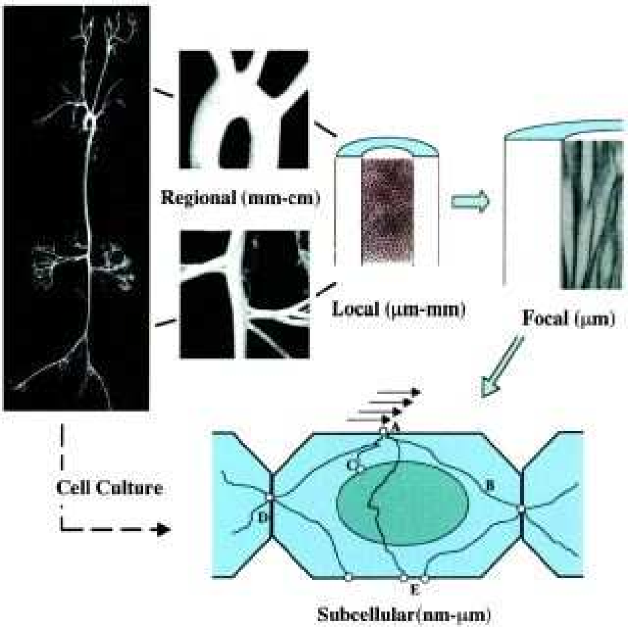
Motivation - Mechanotransduction in blood vessels

- endothelial cells line blood vessels
  continuously exposed to blood flow
- stimulation of a mechanical sensor (fluid-structure interaction)$\star$
  → transmission of stress through sensor
  → stress transduction biochemical signals



Fluid equations for blood flow

- Apply Newton's 2^{nd} law, $\mathbf{F} = \mathbf{ma}$ to the motion of a portion of fluid

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}$$

$$= \nabla \cdot \sigma + \mathbf{f}$$

- conservation of mass

$$\nabla \cdot \mathbf{u} = 0$$

Navier-Stokes equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\nabla\cdot\mathbf{u} = 0$$

- non-dimensionalize with the following changes of variables:

$$\mathbf{u} = U\mathbf{u}' \quad \mathbf{x} = L\mathbf{x}' \quad \mathbf{t} = T\mathbf{t}' \quad \mathbf{f} = F\mathbf{f}' \quad \mathbf{p} = P\mathbf{p}'$$

$$\rho\frac{U\partial\mathbf{u}'}{T\partial t} + \rho\frac{U^2}{L}\mathbf{u}'\cdot\nabla'\mathbf{u}' = -\frac{P}{L}\nabla' p' + \mu\frac{U}{L^2}\triangle'\mathbf{u}' + F\mathbf{f}'$$

$$P = \frac{\mu U}{L} \quad F = \frac{\mu U}{L^2}$$

Stokes equations

$$\frac{L}{UT}\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\frac{1}{Re}\nabla' p' + \frac{1}{Re}\triangle' \mathbf{u}' + \frac{1}{Re}\mathbf{f}'$$

Where Re is the *Reynolds number* and is defined by:

$$Re = \frac{\rho U L}{\mu}$$

Re physically describes the ratio of inertial forces to viscous forces

- If $Re << 1$ and we choose $T = \frac{L}{U}$, we arrive at the steady Stokes equations:

$$\begin{aligned} \triangle \mathbf{u}' &= \nabla p' - \mathbf{f}' \\ \nabla' \cdot \mathbf{u}' &= 0 \end{aligned}$$

Re in blood vessels

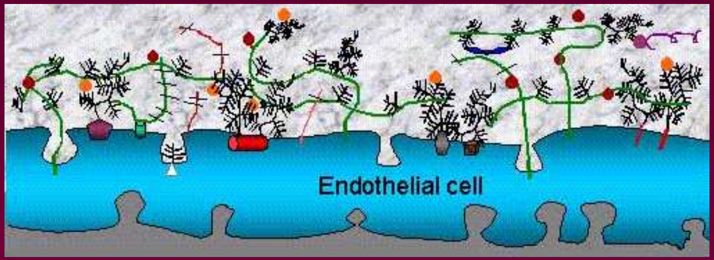
- Re can vary a lot depending on vessel
- capillary:
  characteristic velocity: $.01cm/s$
  characteristic length (vessel diameter): $15\mu m$
  blood viscosity: 3.38x$10^{-2}cm^2/s$

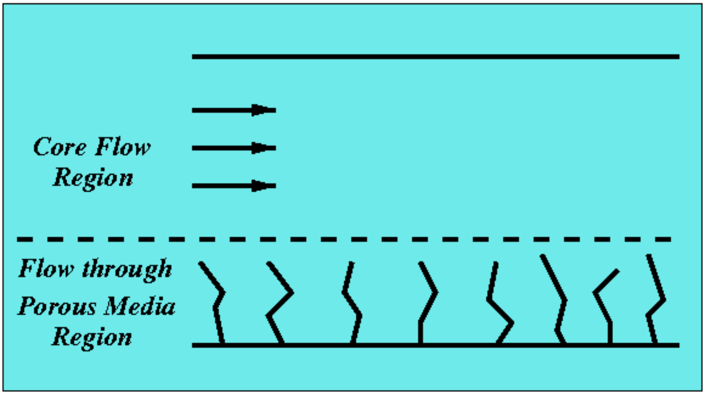
$$Re \approx 8 \times 10^{-5}$$

- aorta:
  characteristic velocity: $21cm/s$
  characteristic length (vessel diameter): $1.13cm$
  blood viscosity: 3.38x$10^{-2}cm^2/s$

$$Re \approx 7 \times 10^{2}$$

Stokes flow through strange geometry





- how can we solve Stokes equations around strange geometry?
- what do we know? Stokes equations are linear and elliptic
- is there a method for a similar problem that can be applied here?

Potential Theory?

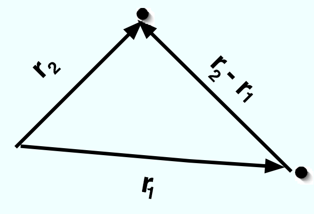
- *Theorem: A function, harmonic and continuously differentiable in a closed regular region R may be represented as the sum of the potentials of a simple and of a double distribution on the boundary of R (Kellogg 1953)*
- What equation has harmonic functions as solutions??
- potential theory → harmonic functions → solutions to Laplace's equation (linear and elliptic) in a bounded domain
- Lets solve the Dirichlet problem of the Laplacian in 3D:

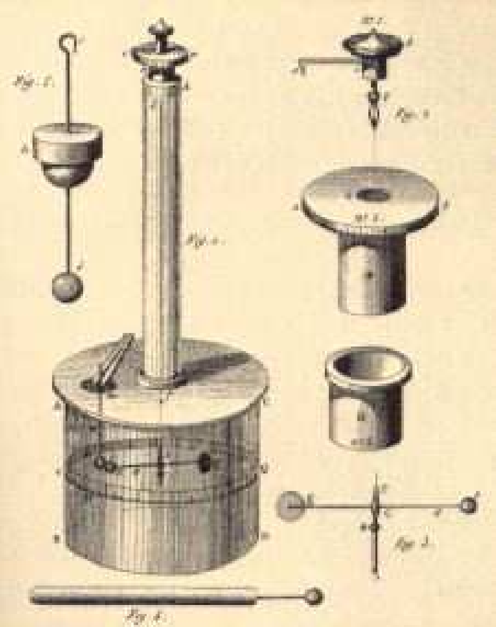
$$\begin{aligned} \triangle u(\mathbf{x}) &= 0 \qquad \mathbf{x} \in \Omega \\ u(\mathbf{x}) &= h(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega \end{aligned}$$

Coulomb’s law 1785

- What is the force on a test charge Q due to a single point charge q?

$$\mathbf{F} = \frac{kQq}{4\pi\epsilon_0}\frac{\mathbf{r_2}-\mathbf{r_1}}{|\mathbf{r_2}-\mathbf{r_1}|^3} = \frac{kQq}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^2}$$





- Force is inversely proportional to the square of the separation distance
- If there are several point charges:

$$\mathbf{F} = Q\mathbf{E} = Qk\sum_{i=1}^{n}\frac{\mathbf{r_i}-\mathbf{r'_i}}{|\mathbf{r_i}-\mathbf{r'_i}|^3}$$

- $\mathbf{E}$ is called the *electric field*

More on Coulomb’s law

For a continuous distribution of volume charge density $\rho(\mathbf{r}')$,

$$\mathbf{E}(\mathbf{r}) = \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'$$

$$= -\int \rho(\mathbf{r}') \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3r'$$

This says that $\mathbf{E}(\mathbf{r})$ is also defined as the gradient of another function

$$\mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r})$$

where $\phi(\mathbf{r})$ is the potential and describes the work done against the electric field to bring a unit charge to $\mathbf{r}$

Gauss’ law

Taking the divergence of each side gives:

$$\nabla \cdot \mathbf{E} = \nabla^2 \phi = \int \rho(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

Gauss’ law (conservation of charge) gives:

$$\int_S \mathbf{E} \cdot \mathbf{n} dS = 4\pi \int_V \rho dV$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

which means

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = 4\pi\delta(\mathbf{r} - \mathbf{r}')$$

Poisson equation - 1812

and so we arrive at Poisson's equation:

$$\nabla^2 \phi = 4\pi\rho$$

since

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) = -\int \rho(\mathbf{r}')\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r'$$

$$\boxed{\phi(\mathbf{r}) = -\int \rho(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3r'}$$

where $\phi(\mathbf{r})$, the potential at $\mathbf{r}$, is the sum of potentials due to a volume distribution of charge

Green’s Functions

- The solution of a differential equation defined over an unbounded region, for a point source of unit strength, is call the fundamental solution of the equation or the free-space Green’s function .
- Let G for the Laplacian be defined as follows:

$$G(\mathbf{x}, \xi) = \frac{1}{4\pi|\mathbf{x} - \xi|}$$

$$\nabla^2 G(\mathbf{x}, \xi) = \delta(\mathbf{x} - \xi)$$

- Physically, can think of G as a potential due to a unit charge at ξ.
- We need more tools to further discuss potentials as the solutions to boundary value problems...

Divergence Theorem

For any n-dimensional domain Ω, with boundary $\partial\Omega$, if F is a vector field and $\mathbf{n}$ is the outward facing normal to the boundary, then

$$\int_{\Omega} \nabla \cdot \mathbf{F} dV = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS - \int_{\mathit{\Omega}} \nabla(\mathit{1}) \cdot \mathbf{F} dV$$

$$= \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS$$

David Bowie says:

If there are no sources or sinks in some bounded region (nothing is created or destroyed), then the density of stuff in that region can only change if stuff leaves or enters by crossing the boundary

Divergence Theorem and Green’s Identities

For any non-singular and at least twice differentiable functions u and v, let $F = v\nabla u$ in the divergence theorem so that:

$$\int_{\partial\Omega} v\nabla u \cdot \mathbf{n} dS = \int_{\Omega} \nabla \cdot (v\nabla u) dV$$

which reduces to Green’s 1st Identity:

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n} dS = \int_{\Omega} (v\triangle u + \nabla v \cdot \nabla u) dV$$

exchanging u for v and subtracting gives Green’s 2nd Identity:

$$\int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) dS = \int_{\Omega} (u\triangle v - v\triangle u) dV$$

Green's Identities and Reciprocal Relation

Looking at Green's second identity again,

$$\int_{\partial\Omega}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS = \int_{\Omega}(u\triangle v - v\triangle u)dV$$

we see that we can again use the divergence theorem to turn the left hand side into a volume integral

$$\int_{\Omega}\nabla\cdot(u\nabla v - v\nabla u)dV = \int_{\Omega}(u\triangle v - v\triangle u)dV$$

$$\nabla\cdot(u\nabla v - v\nabla u) = (u\triangle v - v\triangle u)$$

now if u and v both satisfy Laplace's equation, then we arrive at the reciprocal relation

$$\nabla\cdot(u\nabla v - v\nabla u) = 0$$

Boundary integral equation formulation

Recall: Dirichlet problem of the Laplacian in 3D:

$$\begin{aligned} \triangle u(\mathbf{x}) &= 0 \qquad \mathbf{x} \in \Omega \\ u(\mathbf{x}) &= h(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega \end{aligned}$$

Begin with Green's second identity:

$$\int_{\Omega} (u \triangle v - v \triangle u) dV = \int_{\partial\Omega} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) dS$$

It is tempting to try to replace v with the free space Green's function:

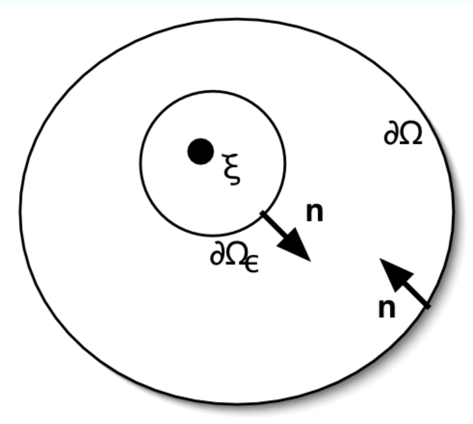
$$\int_{\Omega} (u \triangle G - G \triangle u dV = \int_{\partial\Omega} (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}) dS$$

$$u(\xi) = \int_{\partial\Omega} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}) dS$$

Boundary integral equation formulation continued

Since G is singular at some point ξ in Ω, we must remove an ϵ ball around the singularity and apply Green's second identity to the new region $\Omega \backslash \Omega_\epsilon$ and take the limit as $\epsilon \rightarrow 0$

- control volume is $\Omega \backslash \Omega_\epsilon$
- boundary becomes $\partial\Omega + \partial\Omega_\epsilon$



$$\int_{\Omega \backslash \Omega_\epsilon} (G \triangle u - u \triangle G)\, dV = \int_{\partial\Omega} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n})\, dS + \int_{\partial\Omega_\epsilon} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n})\, dS_\epsilon$$

Boundary integral equation formulation continued

$$\underbrace{\int_{\Omega \backslash \Omega_\epsilon} (G \triangle u - u \triangle G) \, dV}_{} = \int_{\partial\Omega} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}) \, dS + \int_{\partial\Omega_\epsilon} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}) \, dS_\epsilon$$

u and G are both harmonic everywhere so this is 0!!

$$\int_{\partial\Omega} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}) \, dS + \underbrace{\int_{\partial\Omega_\epsilon} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}) \, dS_\epsilon}_{} = 0$$

Let us now just deal with the integral over the ϵ sphere $\partial\Omega_\epsilon$

$$\int_{\partial\Omega_\epsilon} G \frac{\partial u}{\partial n} \, dS_\epsilon = \text{"single layer potential" of density } \frac{\partial u}{\partial n}$$

$$- \int_{\partial\Omega_\epsilon} u \frac{\partial G}{\partial n} \, dS_\epsilon = \text{"double layer potential" of density } u$$

Double Layer Potential

Let $u(\xi)$ be u evaluated at the singularity and $u(s)$ be u on $\partial\Omega$.

$$\int_{\partial\Omega_\epsilon} u(s)\frac{\partial G}{\partial n}dS_\epsilon(s) = \int_{\partial\Omega_\epsilon}[u(s) - u(\xi)]\frac{\partial G}{\partial n}dS_\epsilon(s) + u(\xi)\int_{\partial\Omega_\epsilon}\frac{\partial G}{\partial n}dS_\epsilon(s)$$

The last integral becomes:

$$\lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon}\frac{\partial G}{\partial n}dS_\epsilon(s) = \lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon}\frac{1}{4\pi\epsilon^2}dS_\epsilon(s)$$

$$= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\epsilon^2}\epsilon^2 sin\phi d\phi d\theta = \frac{1}{2}\left[-cos\theta\right]_0^{\pi} = 1$$

since $u(s)\to u(\xi)$ as $\epsilon\to 0$, the first integral on the right hand side becomes 0 and finally we have

$$\lim_{\epsilon\to 0}\int_{\partial\Omega_\epsilon} u(s)\frac{\partial G}{\partial n}dS_\epsilon(s) = u(\xi)$$

Single Layer Potential

$$\lim_{\epsilon \to 0} \int_{\partial\Omega_\epsilon} G \frac{\partial u}{\partial n} dS_\epsilon = \int_0^{2\pi} \int_0^{\pi} \frac{1}{4\pi\epsilon} \frac{\partial u}{\partial n} \epsilon^2 \, sin\phi d\phi$$

as $\epsilon \longrightarrow 0$, this integral will be zero provided $\frac{\partial u}{\partial n}$ is not too large. Putting the pieces together...

$$\lim_{\epsilon \to 0} \int_{\partial\Omega_\epsilon} (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}) dS_\epsilon = -u(\xi)$$

So we finally have the relationship we had hoped for:

$$\boxed{u(\xi) = \int_{\partial\Omega} \left(G(\xi, s) \frac{\partial u(s)}{\partial n} - u(s) \frac{\partial G(\xi, s)}{\partial n} \right) dS}$$

Boundary integral equation formulation

We are trying to solve Laplace's equation on a bounded domain:

$$\begin{aligned} \triangle u(\mathbf{x}) &= 0 \qquad && \mathbf{x} \in \Omega \\ u(\mathbf{x}) &= h(\mathbf{x}) \qquad && \mathbf{x} \in \partial\Omega \end{aligned}$$

We have the following integral representation:

$$u(\xi) = \int_{\partial\Omega} G(\xi, s)\frac{\partial u(s)}{\partial n}\, dS - \int_{\partial\Omega} u(s)\frac{\partial G(\xi, s)}{\partial n}\, dS$$

WHAT IS WRONG WITH THIS REPRESENTATION?

Hint: What is known on the right hand side?

Boundary integral equation

- take the limit as the point ξ goes to a point b on the boundary
- $0 < \theta < \pi$ instead of 2π as before
- take the limit as $\epsilon \rightarrow 0$ and we have

$$\lim_{\epsilon \longrightarrow 0} \int_{\partial\Omega_\epsilon} (G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n})\, dS_\epsilon = -\frac{1}{2}u(b)$$

so we are left with the **boundary integral equation**:

$$\boxed{u(b) = 2\int_{\partial\Omega}\left(G(b,s)\frac{\partial u(s)}{\partial n} - u(s)\frac{\partial G(b,s)}{\partial n}\right) dS}$$

Fredholm integral of the 1^{st} *kind*

Application of boundary integral equations

Stokes Equations - linear $\&$ elliptic

$$\mu \triangle \mathbf{u} = \nabla p - \mathbf{F}$$
$$\nabla \cdot \mathbf{u} = 0$$

Plan of attack to use the boundary integral method:

1. derive Green's identities corresponding to Stokes equations
2. find Green's function(s)
3. develop boundary integral representation in terms of the velocity and surface force

Introduction to Index Notation

- simplify writing multiple equations and matrix/vector components
- Example:

$$a_1x_1 + a_2x_2 + a_3x_3 = \sum_{i=1}^{3} a_ix_i = a_ix_i = p$$

- repetition of an index implies summation *(dummy index)*
- index that occurs once is called a *free index*
- number of free indices indicates order of term:

$a_ix_i = p$ NO free indicies → scalar
$A_{ij}b_j = v_i$ one free index → vector
$u_iv_j = A_{ij}$ two free indicies → matrix

Deriving Green’s Identities for Stokes Flow

- Recall Green’s 1st identity:

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n}\,dS = \int_{\Omega}(v\triangle u + \nabla v\cdot\nabla u)\,dV$$

- we also have the relationship:

$$v(\nabla\cdot\nabla u) = \nabla\cdot v\nabla u - \nabla v\cdot\nabla u$$

- compute $\mathbf{u}'\cdot(\nabla\cdot\sigma)$. In index notation:

$$u'_j\frac{\partial\sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(u'_i\sigma_{ij}) - \sigma_{ij}\frac{\partial u'_i}{\partial x_j}$$

Reciprocal Relation

- using the definition of σ and proceeding as before...
- Stokes flow counterpart to Green's second identity:

$$u'_j \frac{\partial \sigma_{ij}}{\partial x_j} - u_j \frac{\partial \sigma'_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i \sigma'_{ij} - u_j \sigma'_{ij})$$

- If both u and u' with associated σ and σ' satisfy regular flows, the reciprocal relation:

$$\frac{\partial}{\partial x_j}(u_i \sigma'_{ij} - u_j \sigma'_{ij}) = 0$$

Fundamental solutions

- free-space Green’s function for the Stokes equations:

$$\mathbf{S}_{ij}(\mathbf{x},\xi) = -\frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{(x_i-\xi_i)(x_j-\xi_j)}{r^3}\right)$$

$$\mathbf{T}_{ijk}(\mathbf{x},\xi) = -\delta_{ik}p_j(\mathbf{x},\xi) + \frac{\partial S_{ij}(\mathbf{x},\xi)}{\partial x_k} + \frac{\partial S_{kj}(\mathbf{x},\xi)}{\partial x_i}$$

$$\mathbf{P}_j(\mathbf{x},\xi) = \frac{\partial}{\partial x_j}\left(\frac{1}{4\pi r}\right)$$

- instead of $\nabla^2 G = \delta$, we have:

$$\boxed{\frac{\partial\sigma'_{ij}}{\partial x_i} = \frac{\partial p}{\partial x_i} - \nabla^2\mathbf{S}_{ij}(\mathbf{x},\xi) = \delta(\mathbf{x}-\xi)\delta_{ij}}$$

Putting it all together...

- we have the analogous Green's second identity
- we have the free space Green's function
- the integral equation representation:

$$u_j(\xi) = -\int_{\partial\Omega} \sigma_{ik}(\mathbf{x})\mathbf{S}_{ij}(\mathbf{x},\xi)n_k(\mathbf{x})\,dS(\mathbf{x}) + \int_{\partial\Omega} u_i(\mathbf{x})\mathbf{T}_{ijk}(\mathbf{x},\xi)n_k(\mathbf{x})\,dS(\mathbf{x})$$

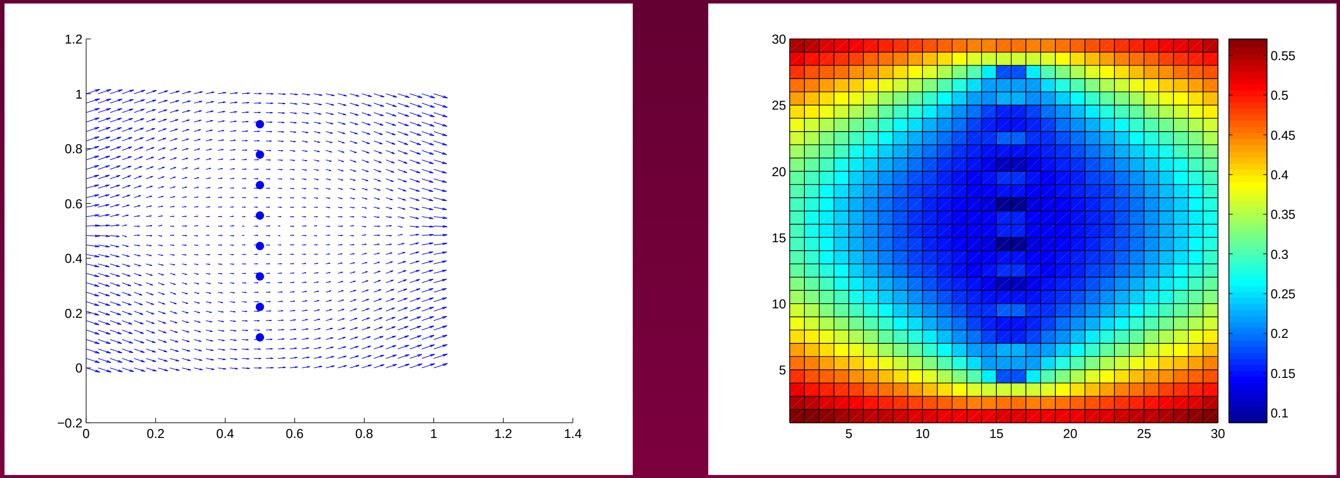
- notice single layer and double layer potentials
- let $\xi \to b$ on the boundary to get the boundary integral equation

$$u_j(b) = -2\int_{\partial\Omega} \sigma_{ik}(\mathbf{x})\mathbf{S}_{ij}(\mathbf{x},b)n_k(\mathbf{x})\,dS(\mathbf{x})$$

$$+2\int_{\partial\Omega} u_i(\mathbf{x})\mathbf{T}_{ijk}(\mathbf{x},b)n_k(\mathbf{x})\,dS(\mathbf{x})$$

Future Project

- method of regularized Stokeslets



- regularized stokeslets $+$ boundary integral equations $=$ FUN???
- THANKS!!!